# Open Dynamical Systems: Ergodic Theory, Probabilistic Methods and Applications

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## 1 Overview of the Field

*Ergodic theory* as a mathematical discipline refers to the analysis of asymptotic or long-range behaviour of a dynamical system (a map or flow on a state space, or more generally a group or semigroup action on a state space) using probabilistic methods. The word 'ergodic' used in this context dates back to the origins of the field in statistical mechanics by reference to the *ergodic hypothesis* (that time averages of observables along orbits be deemed equal space averages) and to the celebrated ergodic theorems of Birkhoff and von Neumann describing conditions under which this property holds. It is a close cousin to topological dynamics, the study of continuous actions on a topological space, and to smooth dynamics, the study of differentiable actions on smooth manifolds. Because of this, it has become quite common in recent years to use the term *measurable dynamics* in place of 'ergodic theory.' There is a rich and productive synergy between the three fields since many interesting examples can be approached through any one, or more typically, a combination of these paradigms. In the case of ergodic theory a key object is an invariant or stationary measure against which observables are integrated to obtain the system statistics and asymptotic behaviour. Determining the properties of such measures is key step for many problems in the field.

Open dynamical systems is currently an active subbranch of dynamical systems research. The difference between an open system and a more traditional 'closed' dynamical system is very simple. In a closed dynamical system, the map (or a flow) maintains its image in the state space for all time, whereas in an open system, orbits may eventually escape from the state space, typically by falling into a 'hole in the space. An everyday example is the dynamics of a billiard ball on a table with a pocket (the hole). Other mechanisms for terminating orbits, for example, diffusion through a boundary or similar stochastic perturbations from a closed system are studied within the broad classification of open dynamics.

The notion of an open dynamical system, and for the moment we focus specifically a map with a hole, was introduced by Pianigiani and Yorke in 1979: [12]. In a series of papers from the 1990s, Collet, Martinez and collaborators developed the framework for a general theory applicable to the case of uniformly hyperbolic Markov maps with a hole. They proved existence, and studied ergodic properties of, the associated conditionally invariant measure, the so-called Yorke-Pianigiani measure (the natural analogue of an invariant measure for a closed system). A representative paper from this corpus would be: Collet, Martinez and Schmitt (1994): [4]. Only slightly later, Chernov, Markarian and collaborators extended the framework to cover a wide range of natural hyperbolic settings in, for example: Chernov, Markarian, and Troubetzkoy (1998): [3]. During this decade, a further extension to include maps that are non-uniformly or weakly hyperbolic has been undertaken (eg. Demers (2005): [5]). A survey of results from this period is contained in Demers and Young (2006): [6]. Thus we observe, in a broad sense, the development of the theory for open systems retracing the evolution of the theory for closed dynamical systems, starting with a few interesting examples, followed by the general case for expanding systems, evolving to hyperbolic systems and finally, more recently to the significant challenge presented by non-uniformly hyperbolic systems.

# 2 Recent Developments

While open systems are extremely natural models for certain applications, there has recently been an emphasis on introducing open ‘components’ in a closed system to study subtle dynamical properties of the closed system. By isolating parts of the phase space, mechanisms for mass transfer between regions or components inside the system, and rates of mixing between the components can sometimes be determined. The key idea is that infrequent transitions leading to slow mixing between quasi-stable states are similar to infrequent escapes from one open component system and subsequent re-injection into another component. A particularly transparent description of this phenomenon is discussed in the recent work of Gonzalez-Tokman, Hunt and Wright (2011): [8] . A selection of papers treating various aspects of current research in open dynamics would also include:

Demers, Wright and Young (2010): [7]

Keller and Liverani (2009): [10]

Bunimovich and Yurchenko (2011): [2]

Bruin, Demers and Melbourne (2010): [1]

The mathematical tools for a rigorous analysis of open and closed dynamical systems cross all three dynamical paradigms: measurable, topological and smooth. These theoretical tools have found applications in non-equilibrium statistical mechanics, molecular dynamics and ‘fluid flow,’ including ocean and atmospheric dynamics. At the same time, the development of numerical schemes for applications, motivated by rigorous theoretical results is an important and active area of research with implications for both theoretical and applied studies.

Finally, another exciting area of research is the development of both theoretical and computational tools for non-autonomous (time varying) systems . This is particularly important for some key applications such as atmospheric or ocean flow modelling, where the time scale of change in the parameters defining the dynamics (temperature, pressure, moisture etc) is on the same order or smaller than the observational time scale to be addressed by the model. Again, there are challenging applications and compelling theoretical questions in this area, for both closed and open dynamical systems.

# 3 Presentation Highlights

A workshop participants gave 35 talks throughout the week, representing all the major threads of pure and applied work in the subject area. The talks can be broadly classified into the following areas. The notation [V] indicates that an online video of the talk is archived on the BIRS website at `http://www.birs.ca/videos/` Search for workshop 12w5050.

## 3.1 Theoretical Developments in Open Dynamical Systems

Mark Demers, Fairfield University. Title: Dispersing billiards with holes, [V]

Carl Dettmann, University of Bristol. Title: Escape and diffusion through small holes

Andrew Ferguson, University of Bristol. Title: The dimension of some sets generated by systems with holes, [V]

Rainer Klages, Queen Mary University of London. Title: Where to place a hole to achieve a maximal diffusion coefficient, [V]

Some of the earliest questions in the field of open dynamical systems involved computing the probability of survival (and related to this the so-called *escape rate* of mass from the system as orbits enter the hole and are terminated). Moving to the finer structure of the system, we may wish to compute the distribution of mass remaining in the system after a fixed number of iterations (suitably renormalized to account for the loss of mass up to that time) and if possible, the limiting distribution as the number of iterations goes to infinity. This limiting distribution is known as the conditionally invariant measure for the open system. Just as in the case of closed systems, the transfer operator (also known as the Ruelle-Perron-Frobenius operator) plays a central role in determining dynamical properties of the open system. For example, the function $f^* \geq 0$ is the density of an absolutely continuous conditionally invariant measure, and $-\log \rho$ is its escape rate iff

$$P_T f^* = \rho f^*,$$

where $0 < \rho < 1$ is the dominant eigenvalue and $f^*$ is the associated eigenfunction for the transfer operator. One well-known approach for finding $f^*$ is through spectral perturbation. Start with the transfer operator for the closed system, then introduce a (small) hole in the space and prove continuity of the spectral projections associated to *peripheral eigenvalues* of the associated open transfer operators; in effect, the invariant density for the closed system (with eigenvalue 1) perturbs to an eigenfunction for $\rho \approx 1$ if the hole is sufficiently small. The key step for making this work is to find an appropriate Banach space of functions, invariant under the action of the transfer operators and for which the transfer operators are, in some sense, uniformly quasi-compact. An important ingredient is the result of Keller-Liverani [11], which provides the theoretical framework over which to build the method.

Ironically, one of the most intuitively attractive open systems, a billiard table with a hole, has proved to be one of the most difficult to treat rigorously with these methods, owing to the prevalence of singular points and unbounded derivative arising from grazing collisions. Demers' presentation described recent joint work with H.K. Zhang, using a new class of anisotropic Banach spaces on which the closed billiard flow, for a wide range of dispersive configurations, allows for a spectral gap in the associate transfer operator. From this it follows that certain open perturbations (=small holes satisfying some reasonable technical assumptions) will admit absolutely continuous conditionally invariant measures. The holes may be on the interior of the table, or in the boundary of the table.

This spectral picture leads another set of natural questions: suppose we study a closed system with a family of holes that 'shrink to a point'. What kind of continuity properties are present for the escape rate and the absolutely continuous conditionally invariant measure? Does the latter converge to an invariant measure, and in what sense? What is the relation between the escape rate and the size of the hole? A rough heuristic goes like this: orbits will follow the closed dynamics until they fall into the hole. By the ergodic theorem, the probability that a typical orbit will fall into the hole should be about the value of the invariant density at the hole, multiplied by the area of the hole. Therefore, up to first order, the 'linear response' of the escape rate to the size of the hole should be the value of the density at the hole. Although this heuristic misses some important points (the obits may fall into the hole long before they conform with the statistics specified by the ergodic theorem, for example) the heuristic is approximately correct. The presentations by Dettmann and Klages address some of these issues, including aspects of the *location* of the hole and higher order estimates of the response function with respect to the hole size.

The *survivor set* is the set of points whose orbit never falls into the hole. This set supports invariant measures for the open system. Motivated by an early number-theoretic investigation by Hensley [9], Ferguson estimates the dimension of the survivor set, compared to the original state space. Both box dimension and Hausdorff dimension were considered and the dependence of each of these on escape rate was discussed. Ferguson's collaborators on this work are T. Jordan and M. Rams.

## 3.2 Applications

Dynamical systems are frequently constructed as models of physical systems; examples of recent interest include ocean and atmospheric flows, trajectories of spacecraft, planetary motion, or models of biological or medical processes. Of course the foundational example in the field is the study of billiards already discussed above. In all of these examples the role of time evolution is clear. Much more subtle use of dynamics has led to important links with probability and stochastic processes, coding theory, aperiodic order (or quasicrystals) and analytic number theory. In these areas, the notion of time evolution is not nearly so transparent.

The following presentations indicate a diverse range of modern applications of ergodic theory to other areas such as physics and engineering.

Kevin Lin, University of Arizona. Title: Reliability of Driven Oscillator Networks, [V]

James Meiss, University of Colorado Boulder. Title: Transport in Transitory Systems: Mixing in Droplets, [V]

Yuzuru Sato, Hokkaido University, Japan. Title: Noise-induced phenomena and their applications, [V]

Paul Tupper, Simon Fraser University. Title: Using the Lorentz gas to resolve a paradox of state-dependent diffusion, [V]

Meiss described an interesting micro-mixing process where small droplets of raw material are placed in a homogenous fluid flow, the goal being to induce laminar mixing of the material inside the droplet. Techniques from the area of non-autonomous flows and coherent structures can be used to study the internal mixing process inside the drop as the drop is transported by the ambient flow through a sinuous channel. This reports on joint work with B. Mosovsky.

Lin address the question of *reliability* in pulse coupled phase oscillator networks by introducing random dynamical systems models. Both stability and instability results were presented.

Tupper discussed an intriguing paradox about a particle moving in a box where there are two disjoint regions with different diffusion constants. The question is, how much time does the particle spend in each part of the box? 'Physicists' have argued that they spend equal amounts of time, whereas mathematical results would suggest more time is spent in the low-diffusion component. By modelling the trajectory by a random Lorenz gas the speaker shows that both conclusions can be attained, depending on the ratio of free volume fractions in the two regions.

T. Sato discussed various noise-induced phenomenon in dynamical systems such as stochastic resonance, noise-induced synchronization and noise-induced chaos. The results of various numerical experiments were presented. This reports on joint work with D. Albers and Y. Tasaka.

## 3.3 Computational Methods in Ergodic Theory

In order to apply the powerful machinery available from ergodic theory to a 'real dynamical system' one first needs to develop numerical schemes to compute or at least estimate one or more of the key quantities arising in the theory. These can include: invariant measures, Lyapunov exponents, entropy, invariant sets and so-on. The desire to develop tools based on ergodic theory are almost as old as the subject itself. For example, already in the 1950's S. Ulam [13] proposed a simple discretization scheme for the transfer operator associated to a map that can be used to compute invariant measures that is still used today in many applications and whose convergence in various settings is still the subject of active research. One should be clear here what the goal is: to develop tractable numerics that can be *proved* to approximate the object or quantity in question, and if possible with *a priori* error bounds. A well-known example is the following. Birkhoff's theorem guarantees that the orbit of almost every point in the phase space will be distributed proportionally to some invariant measure. The problem is that an orbit is an infinite object, difficult or impossible to compute with sufficient precision even when truncated and the qualification of 'almost every' is with respect to an unknown measure (often the one you are trying to compute). So this is an intractable numerical scheme, but none the less one that is frequently used to get a non-rigorous feel for the character of an invariant measure when nothing rigorous is available. Generally, we aim to do better than this.

For closed, discrete time systems, the transfer operator (also called the Perron-Frobenius operator) allows us to study the dynamics acting, not on individual points in the state space, but on functions defined over the state space. For continuous time systems, the infinitesimal generator of the associated group or semigroup action plays a similar role. Estimating dynamical properties is therefore frequently done by studying the related linear operator (transfer or infinitesimal) acting on appropriately chosen Banach spaces of functions over the state space. A similar approach can be applied to open dynamical systems.

The following presentations treated a broad range of approaches and an up-to-date discussion of some of the issues at the forefront of research in numerical ergodic theory.

James Yorke, University of Maryland. Title: Partial Control, [V]

Oscar Bandtlow, Queen Mary University of London Title: Lagrange-Chebyshev approximation of transfer operators, [V]

Rua Murray, University of Canterbury. Title: Numerical approximation of conditionally invariant measures, [V]

Oliver Junge, TU Muńchen. Title: Estimating long term behavior of flows without trajectory integration: An infinitesimal generator approach, [V]

Michael Dellnitz, University of Paderborn. Title: The Computation of Invariant Sets via Newton's Method, [V]

Leonid Bunimovich, Georgia Institute of Technology. Title: Finite-time dynamics, [V]

Erik Bollt, Clarkson University. Title: Basis Markov Systems, Estimated Transfer Operators of Open Systems, and Absorbing Markov Chains, [V]

Robyn Stuart, University of New South Wales. Title: Almost-invariance in Open Dynamical Systems, [V]

Many computational procedures in ergodic theory are based on a Galerkin-type method of projection of the transfer operator to a finite-dimensional subspace. Ulam's method [13] is such an example, where the subspace is piecewise constant functions with respect to some finite partition (usually boxes). Oscar Bandtlow described a method based on projection to polynomials via Lagrange-Chebyshev interpolation. For sufficiently smooth maps, it is possible to prove convergence and establish convergence rates of the Lagrange-Chebyshev approximation for the invariant density. In a similar vein, Erik Bollt introduced the concept of *basis Markov* for a dynamical system, wherein the transfer operator exactly preserves a finite dimensional functional subspace of $L^1$, in much the same way that a Markov partition allows for a SFT representation of the dynamics. Examples related to the Hénon map were given. Finally, Oliver Junge explained how these techniques can be used to study *flows* by using a Galerkin approximation of the infinitesimal generator. This allows one to avoid the computationally costly step of computing long flow trajectories over continuous time. Projections to spaces spanned by piecewise constant, Chebyshev and Fourier bases were each discussed. Oliver's talk presented joint work with G. Froyland and P. Koltai.

Robyn Stuart introduced the notion of *invariance ratio* for a subset of an open dynamical system. Subsets with large invariance ratio escape the system slowly and an upper bound on the invariance ratio is the escape rate for the system. A Markov chain interpretation was given, and using this, a computational method for estimating the maximum invariance ratio was developed. Application to computation of almost invariant sets in closed systems was briefly discussed. This talk was based on joint work with G. Froyland and P. Pollett.

Jim Yorke's presentation described a method to control a dynamical system despite the presence of noise or disturbances. One can imagine this as a kind of dynamical game theory, since the disturbances can be thought of as either random noise or as purposeful, hostile efforts of an opponent. The mathematical problem is to keep trajectories inside some specified region of the phase space despite the disturbances. Yorke showed this is possible in some cases, even when the applied control is constrained to be smaller than the applied disturbance. The method involves computation of a *safe set* in phase space; the idea is to always return to the safe set after each iteration. An algorithm for computing safe sets was described and an example application for the Duffing oscillator was given. The work presented is joint with Juan Sambrano, Samuel Zambrano, and Miguel A. F. Sanjuan.

Rua Murray reported on recent work that applies methods from convex optimization to the problem of computing conditionally invariant measures and escape rates in open systems. A bi-level approach is used, wherein one first specifies the escape rate, then computes an associated conditionally invariant measure for that rate through a moment-constrained maximum entropy problem. This work is joint with C. Bose.

Michael Dellnitz focussed on *invariant sets* and their computation. Various box-subdivision algorithms were described that are frequently used to computed invariant submanifolds such as global attractors. Applications to multi-objective optimization were described. A new set-based method analogous to Newton's method for root finding was described. The technique shows particular promise for cases where the invariant set is *unstable* for the global dynamics. This is joint work with Baier, Hessel-von Molo, Kevrekidis and Sertl.

Leonid Bunimovich discussed a notion of *finite-time* escape dynamics. It is known that the position of the hole typically affects the escape rate, even if the holes have the same size. It turns out that for two different hole positions, it is possible to determine in finite time which has the faster escape rate. This is interesting since the escape rate itself is only determined in the limit as time tends to infinity; still the *order* of escape rates can be compared in finite time. Application to dynamical networks was discussed.

## 3.4 Non-autonomous Dynamical Systems - theory and applications

In some realistic applications, time-varying parameters governing the flow or transformation on the state space necessitate modelling by a non-autonomous system. While the ergodic theory of non-autonomous systems parallels that of autonomous dynamics in many ways, there are important differences. Stable and unstable fibrations, a foundation of geometric analysis for an autonomous map or a flow, are now equivariant, time-dependent structures. Other dynamical objects such as Lyapunov exponents and Oseledets subspaces can be used in alternative ways to describe non-autonomous or random dynamics. Invariant or almost invariant objects arising in autonomous flows have non-autonomous analogues called *coherent structures*, features that move around in the state space under time evolution but that may still represent barriers to mixing and relaxation to 'equilibrium,' a concept that also has to be reinterpreted compared to the autonomous setting.

Sanjeeva Balasuriya, Connecticut College. Title: Flow barriers and flux in non-autonomous flows, [V]

Cecilia González Tokman, University of Victoria. Title: A semi-invertible operator Oseledets theorem, [V]

Simon Lloyd, University of Sao Paulo. Title: Slowly decaying modes for skew-product systems, [V]

William Ott, University of Houston. Title: Memory loss for time-dependent dynamical systems, [V]

Kathrin Padberg-Gehle, TU Dresden. Title: Finite-time entropy: a probabilistic approach for measuring nonlinear stretching, [V]

Shane Ross, Virginia Tech. Title: Geometric and probabilistic descriptions of chaotic phase space transport

Naratip Santitissadeekorn, Clarkson University. Transfer operator approach for finite-time coherent sets identification, [V]

Tom Watson, UNSW Title: Computing Oseledets subspaces: A short overview, [V]

A key theoretical tool for non-autonomous systems is the Oseledets Multiplicative Ergodic Theorem for matrix cocyles. There is both an invertible and a non-invertible version of this classical theorem. At this meeting, an alternative setting wherein the fixed time maps need not be invertible, but the timing sequence or *base* is invertible was discussed. This is the so-called semi-invertible case and an Oseledets-type splitting can still be obtained. Extensions from the matrix cocycle setting to operators on an infinite-dimensional Banach space are important for many applications.

Cecilia González-Tokman presented a semi-invertible multiplicative ergodic theorem that for the first time can be applied to the study of transfer operators associated to the composition of piecewise expanding maps randomly chosen from a set of cardinality of the continuum. This is one possible infinite-dimensional setting alluded to above. This work (joint with A. Quas) extends the range of semi-invertible systems that can be handled theoretically. Naratip Santitissadeekorn discussed the use of variational techniques for the computation of finite-time coherent states with applications to geophysical and atmospheric models. This reports on joint work with G. Froyland and A. Monahan. Simon Lloyd looked at cocycles built from a countable collection of expanding interval maps with a SFT base. This is joint work with G. Froyland and A. Quas.

Tom Watson gave an overview of numerical challenges encountered in computing or estimating the equivariant spaces (the so-called Oseledets subspaces) derived in a semi-invertible setting. This is recent joint work with G. Froyland, T. Huels and G. Morriss.

Sanjeeva Balasuriya discussed time-varying stable and unstable manifolds as *flow barriers* that are related to the boundaries of coherent states. Methods for locating these objects at a given time slice and estimating the flux across them (unlike the autonomous case, these are not true barriers to flow) were presented. An application to the non-autonomous double gyre example was presented.

Shane Ross gave a brief overview of geometric and probabilistic methods that have been successfully applied in the case of autonomous systems, and showed how these can be recast for aperiodic time-dependent or what he called *data-based* models. A brief overview of connections to concepts such as symbolic dynamics, chaos, coherent sets, and optimal control was given, with highlights to some recent applications in areas such as celestial mechanics, musculoskeletal biomechanics, ship capsize prediction, and atmospheric microbe transport.

William Ott introduced the notion of memory loss in dynamical systems as an example of exponential decay of transient behaviour. When the system is non-autonomous, one cannot expect a fixed limiting measure to attract all measures under the action of the flow, but rather there should be a path of time-evolving measures that attract all others under evolution of the flow. Various settings in which this sort of result holds were discussed and the situation for open dynamics was briefly touched on. The main results presented in this talk are joint work with M. Stenlund, L.S. Young, C. Gupta and A. Török.

Kathrin Padberg-Gehle introduced us to a new method for estimating nonlinear *streching* processes in a flow, using what she called finite-time entropy. This provides another tool (along with finite-time Lyapunov exponents, and spectral methods based on approximate transfer operators) for studying mixing in non-autonomous flows. The method is based on estimating the entropy growth of a small, localized density under evolution of the transfer operator. This is joint work with G. Froyland.

## 3.5 Theoretical Developments in Ergodic Theory; Closed Systems

Theoretical research in ergodic theory with applications to important physical and scientific problems continues at a very high rate. The importance of new theoretical results for closed systems to modelling and theoretical progress for open systems should now be clear from the foregoing discussion. In this section we describe a group research talks on theoretical ergodic theory.

Viviane Baladi, Ecole Normale Superieure. Title: On the Whitney-Holder differentiability of the SRB measure in the quadratic family, [V]

Henk Bruin, University of Surrey. Title: Renormalization and Thermodynamic Formalism in Subshifts, [V]

Peyman Eslami, Concordia University. Title: On the acim-stability of piecewise expanding dynamical systems, [V]

Pawel Gora, Concordia University. Title: Random map model of metastable system, [V]

Nicolai Haydn, Southern California. Title: The almost-sure invariant principle for uniformly strong mixing measures, [V]

Carlangelo Liverani, University of Rome Tor Vergata. Title: Partially hyperbolic systems close to trivial extensions, [V]

Ian Melbourne, University of Surrey. Title: Convergence and asymptotics of moments for billiards and Lorentz gases

Matt Nicol, University of Houston. Title: Erdos Renyi limit laws for dynamical systems, [V]

Mark Pollicott, University of Warwick. Title: Asymptotics of geodesic flows, [V]

Dalia Terhesiu, University of Rome, Tor Vergata. Title: A new technique for sharp mixing rates associated with infinite measure preserving systems, [V]

Roland Zweimuëller, University of Vienna. Title: Systems with holes as a tool for recurrent infinite measure systems, [V]

A common theme in ergodic theory is 'stability' of dynamical objects, like invariant measures, with respect to perturbation of the system. Viviane Baladi derived weak differentiability (Whitney-Holder regularity) for the path of Sinai-Ruelle-Bowen (SRB) measures arising from a smooth curve of maps within the quadratic family. This work extends similar results due to Ruelle, for example, (with a stronger notion of regularity) for paths of uniformly hyperbolic maps.

Carlangelo Liverani presented a dynamical model of the heat equation built as a small perturbation of a smooth expanding map of the circle times the identity. Exponential decay of correlation results are obtained with respect to the SRB measure. This is joint work with Dolgopyat and De Simoi.

Henk Bruin reported on the connection between renormalization processes and phase transitions in one-dimensional dynamics. A motivating example of the Thue-Morse substitution and Feigenbaum maps was investigated.

Peyman Eslami presented a new Lastota-Yorke inequality that can be applied to maps with periodic turning points. As an application, he showed how this leads to spectral stability type results for associated open systems. Some of the work presented is joint with M. Misiurewicz and P. Gora.

Pawel Gora presented a random map model for almost invariant states and showed the residence time in the two almost invariant domains can be computed from the 'gate sizes' and the relative probabilities that one passes through a gate.

Invariant measures on shift spaces over finite or infinite alphabets are uniformly strong mixing if they satisfy a weak kind of mixing condition. Nicolai Haydn showed that in this situation the information function satisfies the almost-sure invariance principle. This extends previously known results such as the CLT and the weak invariance principle already known in this context.

Mark Pollicott spoke about geodesic flow on surfaces of negative curvature. Various limit laws on the rate at which orbits flow into a cusp were presented, starting with an early (1982) result of Sullivan and, more recently, an estimate on the maximum displacement up the cusp in time $T$ that turns out to be equivalent to a classical result in continued fractions. In the second half of the talk, a dynamical proof of the convergence of the Schottley-Klein function was sketched.

It has been known for some time that a central limit theorem holds for Axiom A diffeomorphisms and also for uniformly hyperbolic systems with exponentially decaying return tails. Ian Melbourne presented recent work (joint with A. Torok) extending these results to non-uniformly hyperbolic maps with polynomial decay of correlation (i.e., polynomial decay of return tails). In this case the correct normalization is $\sqrt{n \log n}$ for the central limit theorem to hold.

Matt Nicol presented large deviation results of Erdös Renyi type, where averages are computed over blocks of consecutive random variables in a stationary sequence. For i.i.d. random variables a classical result shows that there is a critical block size necessary to obtain a value strictly between zero and the essential supremum of of the random variable. Dynamical versions of the result for maps having large deviation principles with rate functions were presented and illustrated with intermittent maps of Liverani, Saussol, Vaienti type. This talk reports on joint work with M. Denker.

Two speakers addressed systems having *infinite invariant measures* (and no finite invariant measure). Typically, these maps are studied by first *inducing* to a suitable subset where the first return map is well behaved, and has a finite invariant measure, and then 'pulling back' this finite measure through the dynamics. Dalia Terhesiu presented improved estimates on correlation decay rates for these systems using a functional operator renewal theorem (related to the induced map), while Roland Zweimuëlle showed how maps with holes can be used to study asymptotics like distributional limits for infinite measure preserving systems.

## 4 Preparation and publication of Ergodic Theory, Open Systems and Coherent Structures

Prior to the meeting date, the organizers contracted with Springer Verlag for production of a book entitled *Ergodic Theory, Open Systems and Coherent Structures* in order to bring together in one place some of the various threads of these new research covered at this workshop. At the time of writing, we have accepted 10 peer-refereed book chapter submissions for this project, representing some of the best cutting-edge research reported on during the conference. We anticipate publication in late 2013 under the series heading *Proceedings in Mathematics and Statistics*. See `http://www.springer.com/series/8806` for further information.

## 5 Outcome of the Meeting

A key goal of the meeting was to bring together both theoretical and applied researchers at one time, in one place for a free exchange on recent scientific results and current challenges. In this respect, the workshop was a great success, drawing widely from both the applied and theoretical research community. Of the forty-three participants arriving in Banff during the week of April 3, sixteen are internationally recognized for their applied work while twenty-five work in more theoretical/foundational areas of the field. Two participants are famous for outstanding work on both theoretical and applied problems! Twenty six participants arrived from outside of North America, travelling from Europe, Asia, South America, Australia and New Zealand. Seven participants were trainees at the time of the workshop, including graduate students or postdocs.

In terms of tangible progress, we know from the submissions to the springer book publications that a number of projects were initiated and/or substantially influenced through direct interaction between researchers while at the workshop. The organizers also know of at least five researchers who began new projects as a result of their experience during the workshop (in addition to the material submitted to the book publication).

The organizers feel our goal to bring together in a meaningful and scientifically productive way both theoretical and applied researchers in the field, to mix young, innovative researchers with established research leaders (end even the inclusion of some pioneers in the field!) along with significant representation by students and postdocs, was a solid success. This BIRS workshop was widely and enthusiastically supported by an strong international contingent of researchers in the field, and we thank the BIRS organization for their large role in bringing it to fruition.

## 6 Final Thoughts

Near the end of the meeting the organizers asked for feedback from participants at the workshop. We collected some comments that future organizers might want to think about and we take this opportunity to share them.

We scheduled a large number of talks, including some one-hour presentations, between 8:30am and 5:30 pm. This made for some very long, intense days. It was suggested that one long talk in the evening, after dinner, might have allowed for a more relaxing schedule during the day. The organizers took some care to make thematic blocks in the schedule, pairing up talks of a similar type. This was appreciated and commented on by participants.

On Wednesday afternoon the workshop organized four options for field trips. The aim was to have something interesting and appropriate for everyone. However, some participants observed that a single large field trip would have given them a better opportunity to talk with colleagues in an informal setting and without the time pressure of the regular workshop schedule.

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