Renormalization Group Methods for Polymer and Last Passage Percolation Problems

Tom Alberts (California Institute of Technology),
Yuri Bakhtin (Georgia Institute of Technology),
Eric Cator (TU Delft),
Konstantin Khanin (University of Toronto)

July 22, 2012–July 29, 2012

1 Overview of the Field

Our workshop is mostly concentrated on the fields of statistical mechanics and probability theory. Here the classical objects are simple random walk and Brownian motion, and these are well studied and commonly used processes. We will be considering directed polymers [4] and last passage percolation paths [2], which in the appropriate light can be seen as simple random walks moving through non-homogeneous environments with random features. Their study overlaps with and provides insight into many different areas of mathematics, such as random matrix theory [7], random Schrodinger operators [5], tropical geometry [6], stochastic PDE [3], and more.

2 Recent Developments and Open Problems

In the last few years a series of papers [1] have started revealing a conjectural universal structure for 1+1-dimensional polymer and LPP models. The former is described by the continuum random polymer, and the latter by the so-called Airy sheet. The connection between the two is not yet fully understood as it is for their discrete analogues, but it is expected that enlightenment will come shortly. Exploring this structure was the main focus of our workshop.

Current understanding of the universal structure follows exclusively from asymptotic analysis of exact formulas from discrete statistical mechanics models. These formulas are very involved and the asymptotic analysis is difficult. Moreover they are generally not robust to even small perturbations in the type of discrete models under consideration, and even discovering them is not easy. At present one of the major obstacles to more progress is a lack of these exact formulas.

3 Scientific Progress Made

The main goal of our project is to understand connections between continuum directed polymer models and continuum last passage percolation. In discrete settings the relationship between these two models is relatively clear, but proper notions of their continuum versions/scaling limits are now just beginning to emerge.

A full understanding of the continuum objects is important for understanding the universal features and statistics of these models, unencumbered from the non-universal and arbitrary features imposed in consideration of the discrete models. The classical analogy here is the relationship between discrete simple random walk and continuum Brownian motion. Studying the latter reveals the universal properties of this type of random spatial motion without having to work around the peculiarities (lattice type, jump distribution) that the discrete walk can impose.

The focus of our workshop was to study the universality structure in the continuum without relying on exact formulas, and instead try to proceed using renormalization group arguments. This moves the analysis more into the framework of dynamical systems, albeit on a complicated space. The advantage is that the renormalization group ideas are simple, intuitive, and useful for enhancing the conceptual understanding of the problem, and with the current progress in this field we believe there is a high likelihood they can be rigorously applied here.

We discovered a particular example where this program can likely be carried out. Consider the quenched endpoint distribution of a continuous random polymer moving through a continuum space-time random environment. Assuming the polymer has time length one, the endpoint distribution can be represented as a random function of x (its density with respect to Lebesgue measure):

$$f_\beta^W(x) := \frac{1}{\mathcal{Z}_\beta^W} \mathbb{E}_{0\to x}\left[:\!\exp\!: \left\{ \beta \int_0^1 W(t, B_t)\, dt \right\} \right] \varrho(x).$$

Here W is a space-time white noise on $[0,1]\times\mathbb{R}$ (the first coordinate is time, the second space), and the expectation $\mathbb{E}_{0\to x}$ is over Brownian bridge paths from 0 to x. Further ϱ is the standard Gaussian density

$$\varrho(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The random variable $\mathcal{Z}_\beta^W$ is a normalization constant so that f_β^W integrates to 1 on the line. We will mostly be interested in the unnormalized quantity

$$\varrho_\beta^W(x) := \mathcal{Z}_\beta^W f_\beta^W(x).$$

Due to the fact that W is a random field whose distribution is invariant under shifts, it follows that the endpoint distribution is the same as that for regular Brownian motion (i.e. $\varrho(x)$) but perturbed by a random process that is necessarily stationary with respect to shifts. This implies that we may rewrite ϱ_β^W as

$$\varrho_\beta^W(x) = \varrho(x) e^{A_\beta^W(x)}$$

where $A_\beta^W(x)$ is a stationary process on the line whose exponential has mean one. Now since a polymer of length 2 can be written as a scaling of a concatenation of two independent polymers of length 1, it follows that

$$\left(\varrho_\beta^W * \varrho_\beta^{\tilde W}\right)(x) \equiv \sqrt{2}\varrho_{\sqrt{2}\beta}^W(\sqrt{2}x),$$

where $\equiv$ indicates equality in law, $\tilde W$ is an independent copy of the white noise, and $*$ indicates convolution. The various factors of $\sqrt{2}$ on the right hand side are because of scaling properties of the Brownian motion and the white noise.

The latter identity in law holds for the particular choice of the stationary process given by $e^{A_\beta^W(x)}$ inherited from the given f_β^W. But it also can be generalized. Let $A(x)$ be a stationary process on the line whose exponential has mean one. Then it is a straightforward calculation to show that

$$\left(\varrho e^A\right) * \left(\varrho e^{\tilde A}\right)(x) = \sqrt{2}\varrho(\sqrt{2}x) e^{C(\sqrt{2}x)},$$

where $\tilde A$ is an independent copy of A, and on the right hand side C is a new stationary process whose exponential has mean one. Thus this class of stationary processes is closed under the operation of multiplication by ϱ and convolution, after an appropriate scaling of the spatial variable. This gives a simple set of dynamics on the stationary processes which we believe can be studied using renormalization group arguments. The main goal is to prove the existence of a fixed point. Other work to be carried out is to prove that these dynamics have only finitely many unstable directions (hopefully just one) in the space of processes, and every other direction is stable (a stable fixed point is clearly $A = 0$).

4 Outcome of the Meeting

Using the framework described above we were able to make some progress towards understanding the structure of the stable manifold of stationary distributions under the given dynamics. In a very rough sense we were able to understand part of the negative spectrum of the operator described in the last section, and we continue to work towards understanding the positive spectrum. As part of our progress we were able to develop some seemingly new techniques for studying stationary distributions on the line that may ultimately be of interest in other applications.

References

[1] Gideon Amir, Ivan Corwin, and Jeremy Quastel. Probability distribution of the free energy of the continuum directed random polymer in $1 + 1$ dimensions. *Comm. Pure Appl. Math.*, 64(4):466–537, 2011.

[2] Eric A. Cator, Leandro P. R. Pimentel, and Marcio W. A. Souza. Influence of the initial condition in equilibrium last-passage percolation models. *Electron. Commun. Probab.*, 17:no. 7, 7, 2012.

[3] Robert Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. *A minicourse on stochastic partial differential equations*, volume 1962 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha.

[4] Giambattista Giacomin. *Random polymer models*. Imperial College Press, London, 2007.

[5] Eugene Kritchevski, Benedek Valkó, and Bálint Virág. The Scaling Limit of the Critical One-Dimensional Random Schrödinger Operator. *Comm. Math. Phys.*, 314(3):775–806, 2012.

[6] Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald. First steps in tropical geometry. In *Idempotent mathematics and mathematical physics*, volume 377 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 2005.

[7] Benedek Valkó and Bálint Virág. Continuum limits of random matrices and the Brownian carousel. *Invent. Math.*, 177(3):463–508, 2009.