Toric Boij–Söderberg Theory

Christine Berkesch (Duke University)
Daniel Erman (University of Michigan)
Gregory G. Smith (Queen's University)

May 20, 2012 – May 26, 2012

1 Overview of the Field

In commutative algebra, homological methods have traditionally centered around minimal free resolutions for modules over a local ring or for graded modules over a standard graded polynomial ring $S := \Bbbk[x_0, \ldots, x_n]$. The key foundational results include the following:

- the Hilbert–Burch Theorem [E, Theorem 20.15], which characterizes the Cohen–Macaulay codimension 2 quotients of S in terms of their minimal graded free resolution;
- the Hilbert Syzygy Theorem [E, Theorem 1.13], which establishes that, for every finitely generated graded S-module, the graded minimal free resolution has finite length (at most $n + 1$); and
- the Auslander–Buchsbaum Formula [E, Theorem 19.9], which relates the depth of a graded S-module to the length of its minimal graded free resolution.

Through the fundamental relation between graded modules over S and sheaves over $\mathbb{P}^n$, all of these results lead to significant insights into the geometry of projective space. Where are the analogous homological tools from commutative algebra for studying the geometry of a smooth toric variety X other than $\mathbb{P}^n$?

The work of David Cox [C] unquestionably provides the correct context. The Cox ring R of X (also known as the total coordinate ring of X) is a positively $\mathrm{Pic}(X)$-graded polynomial ring. The fundamental relation extends to a beautiful correspondence between $\mathrm{Pic}(X)$-graded modules over R and sheaves on X. Unfortunately, the minimal $\mathrm{Pic}(X)$-graded free resolutions for R-modules do *not* provide similar geometric insights on X; see [MS, §7]. For example, the length of a minimal $\mathrm{Pic}(X)$-graded free resolution can be significantly larger than the dimension of X. Roughly speaking, the geometric information is concealed by extraneous algebraic strands in the resolution. How should one extract the significant data from a minimal $\mathrm{Pic}(X)$-graded free resolution?

2 Recent Developments and Open Problems

The most spectacular recent development in the homological methods for S is known as Boij–Söderberg Theory. Born out of the 2006 conjectures of Mats Boij and Jonas Söderberg [BS], this theory describes an unexpectedly simple polyhedral structure on the Betti numbers of graded S-modules. While proving these conjectures, David Eisenbud and Frank-Olaf Schreyer [ES] also discovered a duality with the cohomology tables of algebraic vector bundles on $\mathbb{P}^n$.

The ambitious goal for this workshop was to create a version of Boij–Söderberg theory relating sheaf cohomology on a toric variety X to appropriate free complexes over the associated Cox ring R. The recent preprint of David Eisenbud and Daniel Erman [E^2], which provides a more general construction for the duality pairing in Boij–Söderberg Theory on $\mathbb{P}^n$, served as the starting point. In particular, we focused on developing the foundational homological results for free complexes with irrelevant homology.

3 Scientific Progress Made

During our week at BIRS, we outlined a broad framework for homological commutative algebra over the Cox ring R. The central objects, which we baptized ***splendid complexes***, are finite free complexes over R with irrelevant homology. By creating and implementing new routines in *Macaulay2* [M2], we explored numerous examples of "short" splendid complexes. Our most important advance was likely the invention of a method for shrinking a splendid complex — we create a smaller free complex by identifying and then removing an irrelevant strand of the complex. By exhaustively iterating this process, we obtained minimal splendid complexes in many different examples.

Building on this method, we formulated precise conjectural analogues for both the Hilbert–Burch Theorem and the Hilbert Syzygy Theorem, and we identified some of the crucial ingredients for a toric variant of the Auslander–Buchsbaum Formula. In addition, we generated a detailed sketch for a proof of our splendid version of the Hilbert–Burch Theorem. We also produced several splendid complexes which we think are new candidates to appear as the extremal rays in a multigraded Boij–Söderberg Theory.

4 Outcome of the Meeting

We now have a clear program for developing the necessary homological techniques in commutative algebra for studying smooth toric varieties. We are actively pursuing the conjectures arising from this workshop. We also expect to transform our *Macaulay2* code into a package that can be used by other researchers and distributed with a future version of this software system. Once we are armed with the required homological tools, we will return to the search for a multigraded decomposition result that parallels that main theorems in the Boij–Söderberg theory for $\mathbb{P}^n$.

References

[BS] Mats Boij and Jonas Söderberg, Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture, *J. Lond. Math. Soc. (2)* **78** (2008), 85–106.

[C] David A. Cox, The homogeneous coordinate ring of a toric variety, *J. Algebraic Geom.* **4** (1995), 17–50.

[E] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics 150, Springer-Verlag, New York, 1995.

[E^2] David Eisenbud and Daniel Erman, Categorified duality in Boij-Sderberg Theory and invariants of free complexes, available at `arXiv:1205.0449v1 [math.AC]`.

[ES] David Eisenbud and Frank-Olaf Schreyer, Betti numbers of graded modules and cohomology of vector bundles, *J. Amer. Math. Soc.* **22** (2009), 859–888.

[M2] Daniel R. Grayson and Michael E. Stillman, *Macaulay2*, a software system for research in algebraic geometry, available at `http://www.math.uiuc.edu/Macaulay2/`

[MS] Diane Maclagan and Gregory G. Smith, Multigraded Castelnuovo-Mumford regularity, *J. Reine Angew. Math.* **571** (2004), 179–212.