Alberta Number Theory Days

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1 Overview

The fourth installment of the Alberta Number Theory Days was held in BIRS on the weekend of June 17 to 19. 29 people participated in this event, including 12 faculty members, 13 graduate students, and 4 postdoctoral fellows. A total of nine 45-minutes lectures were given. Three of the lectures were delivered by postdoctoral fellows from Alberta and British Columbia and the remaining six lectures were given by professors from Alberta.

The nine lectures covered a wide variety of areas in number theory. The break down of the lectures in terms of topics was as follows:

1. Analytic Number Theory, three lectures
2. Algebraic Number Theory, two lectures
3. Diophantine Approximation, one lecture
4. Computational Number Theory, two lectures
5. Representation Theory of p-adic Groups, one lecture

2 Summary of Lectures

Nathan Ng started his talk by asking the question "what is an L-function?". He then described a possible answer to this important question given by Selberg [14] in 1989 at Amalfi's conference. The so called "Selberg class" is a family of Dirichlet series that satisfy certain analytic properties (analytic continuation, functional equation, Euler product, and Ramanujan bounds) similar to the classical Riemann zeta function. After describing the concept of a primitive element of the Selberg class, Ng stated the conjecture that all non-real zeros of any primitive element in the Selberg class are simple. After reviewing the known results of Levinson [7] and P. Bauer [1] on this conjecture for the degree one L-functions, Ng discussed his recent joint work with Milinovich where they investigated simple zeros of degree two L-functions (modular L-functions).

Habiba Kadiri discussed an explicit estimation on the error term in the celebrated Prime Number Theorem. After reviewing the classical method due to Rosser and Schoenfeld [10], [11] on this problem, she explained her recent work with Faber where they use smooth functions and an explicit estimate for the density of zeros of the Riemann zeta function to generalize and improve upon the previous results.

Tim Trudgian gave a lively talk on Skewes' Number. The Prime Number Theorem asserts that $\pi(x)$, the number of primes not exceeding x, is asymptotic to $\mathrm{li}(x)$, the logarithmic integral. It is known that, for all $2 \leq x \leq 10^{14}$, the inequality $\pi(x) < \mathrm{li}(x)$ is true. That there are infinitely many x for which this inequality does not hold was proved by Littlewood in 1914. In 1954 Skewes [16] proved that the inequality

$\pi(x) < \mathrm{li}(x)$ will be violated for a number less than $10^{10^{10^{10^3}}}$. In his talk Trudgian gave an account of the work done on reducing this bound. The latest result on this problem is due to Demichel and Saouter [12] where they reduced the Skewes bound to $1.397116701 \times 10^{316}$. Finally, after describing Lehman's work on this problem, Trudgian commented on the possibility of reducing Lehman's bound [6] by employing the recent result [17] of himself on a sharp upper bound for the function $S(T)$, the argument of the Riemann zeta-function along the critical line.

Al Weiss's talk entitled "What do Artin L-functions know about Galois module structure?" gave an account of Weiss's impressive work (joint with Gruenberg and Ritter) ([5], [8], [9]) on the so called "lifted root number conjecture". This conjecture, which is stronger than Chinburg's root number conjecture, is related to the Galois module structure of S-units of a number field. It is also related to Stark's conjecture which can be considered as a generalization of the analytic class number formula to Artin L-functions.

By describing Stark's conjecture and the lifted root number conjecture for a special class of number fields, **Paul Buckingham**'s talk nicely complemented Al Weiss's lecture. After giving an overview of these conjectures and describing them in a baby example of real quadratic fields, Buckingham provided a description of his work on these conjectures for multiquadratic extensions of an arbitrary number field.

A polynomial-exponential Diophantine equation is an equation in the form $f(x) = y^n$ where $f(x)$ is a polynomial with integer coefficients and y is a fixed positive integer. **Mark Bauer**'s talk was centered around finding the solutions of such equations (i.e. pairs (x, n) where x and n are both integers). He explained that while many Diophantine equations can be attacked using variants of Wiles' approach in proving Fermat's Last Theorem [18], polynomial-exponential Diophantine equations do not usually yield to such techniques. In this lecture Bauer reported on his joint work with Mike Bennett where they use restricted (and unrestricted) irrationality measures to find all possible solutions of equations of the type $x^3 + D = y^n$, where n is in the form $3k + 1$.

Renate Scheidler's lecture was on "Infrastructure of Function Fields". The infrastructure concept was proposed by Shanks in 1972 in the case of indefinite binary quadratic forms and later was further explored by Williams for the ideals of real quadratic fields and orders (see [15] and [2]). The infrastructure is a mathematical system that "just barely" fails associativity; however it is suitable for the baby step giant step framework found in many algorithms of computational number theory. In her lecture, Scheidler described her work [13] and the recent work of Adrian Tang on the extension of this important framework to function fields.

An important application of infrastructure is in the computing of the class number and regulators of quadratic fields. This was the subject of talk by **Michael Jacobson**, where he discussed recent efforts to extend existing, unconditionally correct tables of both imaginary and real quadratic fields. Such tables are used to provide valuable numerical evidence in support of a number of unproven heuristics and conjectures, including those due to Cohen and Lenstra [3], [4]. Jacobson also described an unconditional verification algorithm based on ideas of Booker (that verifies the truth of the table generated under the assumption of the Riemann hypothesis) which surprisingly uses the trace formula of Maass forms.

Masoud Kamgarpour lectured on his joint work with Clifton Cunningham on Geometrization of characters of the multiplicative group of local fields. He described in several concrete example (such as the case of multiplicative group and the truncated Witt ring over the finite field of p elements) how one can consider sheaves on these algebraic varieties as proper generalizations of characters of local fields of characteristic zero.

3 Concluding Remarks

A number of participants commented that Alberta Number Theory Days 2011 was a success. It helped to strengthen the bonds (both academic and personal) and forge new links between number theorists in Alberta. The meeting allowed number theorists working in a wide variety of areas to share knowledge and discuss recent progress in their fields.

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