Extending Properties of Tournaments to k-Traceable Oriented Graphs (11frg171)

Ortrud R. Oellermann (University of Winnipeg),
Susan van Aardt (University of South Africa),
Jean E Dunbar (Converse College),
Marietjie Frick (University of South Africa), and
Morten Hegner Nielsen (Thompson Rivers University)

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1 Overview of the Field, Recent Developments and Open Problems

A graph or digraph is *hamiltonian* if it contains a cycle that visits every vertex, and *traceable* if it contains a path that visits every vertex. A (di)graph is *k-traceable* if each of its induced subdigraphs of order k is traceable. A digraph D is *strong* if for every pair u, v of vertices in D there is a directed path from u to v and a directed path from v to u.

A digraph D obtained by assigning directions to the edges of a graph G is called an *oriented graph*. We say that D is an orientation of G. A *tournament* is an orientation of a complete graph and a *multipartite tournament* is an orientation of a complete multipartite graph.

Our interest in k-traceable oriented graphs stems from the following conjecture, which is stated in [2].

The Traceability Conjecture (TC): *For $k \geq 2$, every k-traceable oriented graph of order at least $2k-1$ is traceable.*

The TC was motivated by the OPPC, an oriented version of the Path Partition Conjecture, which can be formulated as follows.

OPPC: *If D is an oriented graph with no path of order greater than λ and a is a positive integer such that $a < \lambda$, then $V(D)$ contains a set A such that the oriented graph induced by A has no path of order greater than a and $D - A$ has no path of order greater than $\lambda - a$.*

If the TC is true, it would imply that the OPPC is true for every oriented graph whose order is exactly one more than the order of its longest paths.

Apart from their connection to the OPPC, k-traceable oriented graphs are a natural generalization of the well-studied class of tournaments, these being the 2-traceable oriented graphs. As such they are of interest in their own right. It is thus natural to investigate to what extent the properties possessed by tournaments extend to k-traceable oriented graphs.

Moon [6] showed that strong tournaments have a very rich cycle structure. In particular he showed that every strong tournament is *vertex-pancyclic*, i.e., in every strong tournament on $n \geq 3$ vertices every vertex belongs to a cycle of length l for every $3 \leq l \leq n$. Thus all strong tournaments have a hamiltonian cycle.

One generalization of tournaments are the multipartite tournaments. The cycle structure of strong multipartite tournaments has been studied by several authors. For example, it is shown by Goddard and Oellermann [4] that every vertex of a strong p-partite tournament D belongs to a cycle that contains vertices from exactly

m partite sets for each m, $3 \leq m \leq p$. Moreover, it is shown by Guo, Pinkernell and Volkmann [5] that if v is a vertex in a strong p-partite tournament, then v lies on some longest cycle and if $p \geq 3$, then v belongs to an m or $(m+1)$-cycle for every m, $3 \leq m \leq p$. More results on paths and cycles in multipartite tournaments appear in an extensive survey of Volkmann [7].

For small values of k the k-traceable oriented graphs share many properties that tournaments possess. For example, it is well-known that every strong tournament is hamiltonian. In [2] it is shown that strong k-traceable oriented graphs are hamiltonian for $k = 3$ and 4. However, when $k \geq 5$ the situation changes dramatically. It was shown in [2] that for every $n \geq 5$ there exists a strong nonhamiltonian oriented graph of order n that is k-traceable for every $k \in \{5, \ldots, n\}$. Thus for $k \in \{2, 3, 4\}$ there are no strong nonhamiltonian k-traceable oriented graphs of order greater than k, while for each $k \geq 5$ there are infinitely many.

Let n and t be integers such that $3 \leq t \leq n$. We say that a digraph D of order n is t-pancyclic if D contains a cycle of length r for every r, $t \leq r \leq n$ and it is *vertex-t-pancyclic* if every vertex is contained in a cycle of every length r for $t \leq r \leq n$. Moreover, D is *weakly (vertex-)pancyclic* if it contains cycles of every length from $g(D)$ to $c(D)$ (through each vertex). It was shown in [1] that for $k = 2, 3, 4$ all strong k-traceable oriented graphs of order at least $k + 1$ are vertex-$(k + 1)$-pancyclic. In the same paper it was observed that this results does not extend to k-traceable graphs for $k \geq 5$. Nevertheless strong k-traceable oriented graphs with girth at least k are girth-pancyclic. The *cycle spectrum* of a (di)graph is the set of the lengths of the cycles in the (di)graph.

Question 1: Is a strong k-traceable oriented graph weakly pancyclic?

Question 2: If the answer to Question 2 is negative is it true that if D is a strong k-traceable oriented graph and m is an integer with $g(D) \leq m \leq c(D)$ that there is D has a cycle of length m or $m + 1$?

Question 3: What is the cycle spectrum of a strong k-traceable oriented graph?

A graph is G is *locally* k-traceable if for every vertex v and every set S of k vertices in $N[v]$, $\langle S \rangle$ is traceable. Every k-traceable graph is obviously locally k-traceable, but not vice versa, since a k-traceable graph of order n has minimum degree at least $n-k+1$, while local k-traceability does not require a minimum degree condition.

Question 4: Under what conditions are locally k-traceable graphs of order n, satisfying the minimum degree condition $\delta(G) \geq n - k + 1$, also k-traceable?

2 Outcome of the Focussed Research Workshop

Questions 1 and 2 were answered in the negative. For Question 3 it was shown that for every pair of integers n, k such that $5 \leq k < n$ and each subset S of $\{n - k + 4, \ldots, n\}$, there exists a strong k-traceable oriented graph of order n with cycle spectrum $\{3, \ldots, n\} - S$.

For Question 4 it was shown that if G is a locally k-traceable graph of order $n \geq k^2 - 5k + 9$ and $\delta(G) \geq n - k + 1$, then G is k-traceable. The requirement on n cannot be relaxed because, for every $k \geq 4$, there exists a locally k-traceable graph of order $k^2 - 5k + 8$ that is not k-traceable.

References

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