REPORT ON BIRS WORKSHOP: "COMPLEX MONGE-AMPÈRE EQUATIONS"

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1. Overview of the field

Complex Monge-Ampère equations occupy a position of central importance in complex geometry and the theory of non-linear partial differential equations. On the geometric side, it has been known since the works of Calabi, Chern, Nirenberg, Yau, Kohn, Fefferman, Siu, and others that the various problems of finding a representative of a cohomology class with given volume form; of finding Kähler-Einstein metrics; and of determining the Bergman kernel and the boundary behavior of holomorphic functions, can all be reduced to the study of a complex Monge-Ampère equation. On the analytic side, complex Monge-Ampère equations are prime examples of fully non-linear elliptic or degenerate-elliptic partial differential equations. They have been extensively investigated, starting with the foundational works of Yau (1978) on a priori estimates on compact Kähler manifolds, of Bedford-Taylor (1978) on generalized solutions, and of Caffarelli-Kohn-Nirenberg-Spruck on boundary regularity for strongly pseudoconvex domains in $\mathbb{C}^n$ (1984).

The last decade has witnessed an explosive growth in the subject, which has opened up entire new venues for investigation. This growth is due on one hand to the unexpected appearance of the Monge-Ampère equation in several new geometric problems (such as the problem of geodesics in the space of Kähler potentials), as well as the need, in both geometry and physics, for considering new related equations (such as the Kähler-Ricci flow, singular Kähler-Einstein metrics, the equation for Kähler metrics of constant scalar curvature, the Strominger system, the Einstein-Sasaki system, and symplectic or even general Hermitian manifolds). On the other hand, there has also been spectacular progress in the development of methods for solving these equations. In particular, powerful new techniques of pluripotential theory, of geometric

flows, of variational methods, and of constructions of solutions by algebraic approximations have been introduced. Some of this is described in greater detail below.

2. The Workshop

The purpose of the workshop was to bring together leading experts in order to discuss these developments and outline open problems. The workshop was timely, since the bulk of the progress described above actually took place in the last three or four years. At the same time, it is perhaps remarkable that major contributions came from researchers from all over the world (Canada, China, France, Great Britain, Poland, Sweden, USA, etc.). Thus the workshop also provided a unique opportunity for interaction between different groups who would normally reside in several distinct continents. With a view towards the long-term vitality of the field, a high proportion of young post-doctoral researchers and graduate students was also invited to participate.

In order to leave a maximum amount of time for discussions and foster close interactions between the participants, the number of formal presentations was kept at 2 per session. The essential role of the formal presentations was to initiate topics of discussions, which can then be taken up at greater length in the more informal sessions outside of the talks.

3. Recent developments discussed at the workshop

The workshop provided an in-depth coverage of the developments described in the overview. These developments are all closely inter-related, but for the purpose of listing them, it is convenient to give the following rough classification. As in the previous discussion, we divide them into groups by either underlying geometric problems, or by method of solution. To highlight the contributions of the participants of the workshop, we have given their names in italics.

(A) Geometric problems

• *Kähler-Einstein metrics:* Perhaps the most famous problem in Kähler geometry is the problem of finding when Kähler-Einstein metrics exist on Fano manifolds. A well-known conjecture of Yau states that the existence of such metrics should be equivalent to the stability of the manifold in the sense of geometric invariant theory. Since Kähler-Einstein metrics can also be viewed as the stationary points

of the Kähler-Ricci flow, this problem is the same as the one of convergence of this flow. Its relation with the Monge-Ampère equation is particularly strong, since the Kähler-Ricci flow is just a parabolic version of the Monge-Ampère equation.

Several approaches to this problem were discussed at the workshop. In particular, in their talks, *Szekelyhidi* discussed the relation of the convergence of the Kähler-Ricci flow to stability conditions such as K-stability, *Berman* discussed a new variational approach to the corresponding Monge-Ampère equation, and *Lu* discussed the notion of K-stability for hypersurfaces.

• *Kähler metrics of constant scalar curvature:* The Kähler-Einstein problem can be viewed as a special case of the more general problem of determining when a positive integral class $c_1(L)$ admits a metric of constant scalar curvature (the Kähler-Einstein case corresponds then to L being the anti-canonical bundle). The conjecture of Yau can be generalized to this case, and the prime candidate for the appropriate notion of stability is K-stability, different versions of which had been defined by Tian and Donaldson. In his talk, *Apostolov* described the construction of constant scalar curvature metrics for manifolds which can be realized as certain fibrations of toric varieties. In his, *Lu* provided an explicit analysis of versions of the notion of K-stability for hypersurfaces.

• *Donaldson's infinite-dimensional geometric invariant theory (GIT):* In the late 1990's, Donaldson proposed another condition for the existence of Kähler-Einstein metrics. This condition can be viewed as an infinite-dimensional version of stability in geometric invariant theory, where geodesics in the space of Kähler potentials play the role of one-parameter subgroups. These geodesics can be interpreted as solutions of a Dirichlet problem for a completely degenerate Monge-Ampère equation, and they are partly responsible for the great recent interest in the solution and properties of such equations. In his talk, *Sturm* described the construction of geodesics as limits of one-parameter subgroups and their resulting regularity. Such results are expected to play a major role in an eventual link of infinite-dimensional GIT with finite-dimensional GIT, as well as in Donaldson's program for the existence of constant scalar curvature metrics.

• *Analytic minimal model program:* When the first Chern class of the manifold is not definite, a smooth Kähler-Einstein metric cannot exist. It is then of great interest to determine which canonical metrics can exist instead, of which singular Kähler-Einstein metrics are

prime examples. There has been considerable progress on this question, thanks in particular to works of Eyssidieux-*Guedj*-Zeriahi, *Demailly*-Pali, Tian-Zhang, *Song*-Tian, and others, making use of the C^0 estimates of *Kołodziej* for the Monge-Ampère equation with L^p right hand sides for $p > 1$. Closely related to such questions are the singularities of the Kähler-Ricci flow on such manifolds. In his talk, *Song* described a complete analysis of the Kähler-Ricci flow on Hirzebruch surfaces. He also outlined what may be viewed as an analytic, Kähler-Ricci flow version of the minimal model program. In his talk, *LaNave* discussed relations of the Kähler-Ricci flow with the moment map, and constructions of test configurations with smooth total space.

• *Extensions of Kähler metrics:* The original Monge-Ampère equation $(\omega + \frac{i}{2}\partial\bar{\partial}\phi)^n = F\,\omega^n$ in Kähler geometry involved a background form ω which is a Kähler form (i.e., closed and positive definite). Applications in modern complex geometry require generalizations in several different directions: when the class of ω is just "big" (*Berman, Demailly*), when the form ω is just a Hermitian form (*B. Guan-Q. Li, Tosatti*-Weinkove, *Dinew-Kołodziej*), or when the underlying almost-complex structure is no longer integrable (Donaldson, *Tosatti*-Weinkove-*Yau*). In their talks, *Q. Li* and *Tosatti* described the solution of the Monge-Ampère equation with Hermitian backgrounds and several geometric applications.

• *Extensions of Calabi-Yau manifolds and Hermitian-Einstein bundles:* Compactifications of superstrings preserving supersymmetry are of great importance in string theory and related areas of theoretical physics. The best-known examples of such compactifications are Calabi-Yau manifolds, the study of which has had a great influence on mathematics in the last 25 years. However, more general compactifications allowing torsion are also of great interest in physics and geometry. They satisfy the so-called Strominger system of equations, which can be viewed as an extension of the Calabi-Yau equation, incorporating torsion as well as the Hermitian-Einstein equation. In his talk, *Fu* described recent joint work of his with Yau, providing a first non-perturbative solution of the Strominger system. Extensions of equations from Kähler geometry to the more general context of balanced metrics were also discussed in his talk as well as in *Yau*'s talk.

• *Sasaki-Einstein manifolds:* Sasaki-Einstein manifolds can be viewed as odd-dimensional analogues of Kähler-Einstein manifolds. They also

arise in compactifications of superstring theory. Although their mathematical theory can be traced farther back, they are still not well-understood. In his talk, *Yau* raised many questions about their existence and properties, and in particular with the formulation of suitable stability conditions.

• *Asymptotic volumes:* The notion of volume is one of the most important geometric invariants associated with a holomorphic line bundle. It is defined by the asymptotics of the dimension of the space of holomorphic sections of the bundle. As such, it admits natural extensions by considering higher cohomology groups. In his talk, *Demailly* described what is presently known about volumes for pseudo-effective bundles, and formulated some precise conjectures. Some key tools in the approaches which he described are approximations of plurisubharmonic functions, the regularity of envelopes of big cohomology classes (joint work of his with *Berman*), and holomorphic Morse inequalities.

• It is well-known, from e.g. the works of Kerzman, Kohn, and Nirenberg in the mid 1970's, that the Monge-Ampère equation can provide some deep information on the boundary behavior of holomorphic functions. In his talk, *S.Y. Li* described its relations with CR geometry and applications. In his talk, *B. Guan* provided a survey of several recent major advances, including the use of subsolutions for the Dirichlet problem instead of pseudoconvexity conditions, the solution of *P.F. Guan* of the Chern-Levine-Nirenberg conjecture, and works of his and *Błocki* on the pluri-Green's function.

(B) Analytic methods

Closely intertwined with the above geometric problems are an array of significant progresses on the analytic methods for solving the Monge-Ampère equation.

• *Pluripotential theory:* In 1998, *Kołodziej* had obtained C^0 a priori estimates for the Calabi-Yau equation, assuming only some integral conditions on the right hand side (for example, L^p integrability for any $p > 1$ is sufficient). Recently, it was shown by *Guedj-Kołodziej*-Zeriahi that the solution is actually Hölder continuous. In his talk, *Dinew* described very recent work of his and co-authors with an explicit and improved Hölder exponent.

• *Approximations by polynomials:* As had been mentioned above, geodesics in the space of Kähler potentials are solutions of a completely degenerate complex Monge-Ampère equation. For the ultimate purpose of relating them to geometric invariant theory, it is of particular interest

not just to construct them, but also to approximate them by geodesics in the finite-dimensional space of Bergman metrics. Such results were described by *Sturm* in his talk. Some key tools are the Tian-*Yau*-Zelditch approximation theorem, Bedford-*Taylor* pluripotential theory, and C^1 estimates of *Błocki, P. Guan, B. Guan-Q. Li.* A precise analysis of the special case of toric varieties has been carried out by *Song* and Zelditch. Here the methods are those of semi-classical analysis and large deviation theory.

• *Variational methods:* Much recent progress on the complex Monge-Ampère equation has been through either a priori estimates and/or pluripotential theory. In his talk, *Berman* described very recent joint works of his with Boucksom-*Guedj*-Zeriahi, where a variational method is developed. Besides providing new proofs of some classical results, it also allowed to extend the theory to big cohomology classes. Interestingly, the method makes use of the sharp form of a Moser-Trudinger inequality established a few years ago by *Phong-Song-Sturm*-Weinkove.

• *Parabolic Monge-Ampère equations:* Another approach to solving the Monge-Ampère equation is through the Kähler-Ricci flow. Here the problem becomes that of determining the time of existence, the singularities which may form, the continuation through the singularities, and the convergence of the flow. In his talk, *Szekelyhidi* considered the Fano case, where the flow exists smoothly for all time and he showed, under various additional assumptions, how K-stability can be used to show the convergence of the flow. Even with the additional assumptions, these results are of particular interest since there are few, if any, results in the direction of the sufficiency of K-stability. In his talk, *Song* described the Kähler-Ricci flow from rough initial data, and how such results can be used to continue the flow through singularities when the first Chern class is not definite.

(C) Open Problems

There was a strong emphasis on open problems at the workshop. Some specific ones which were discussed extensively were the following:

• The talk of *Yau* was devoted almost entirely to the description of open problems ranging from the mid 1970's to present day. They span the entire breadth of Kähler geometry, complex Monge-Ampère equations, and their extensions and applications to string theory and theoretical physics. They include affine Monge-Ampère equations, the Strominger-Yau-Zaslow conjecture, the Strominger system, Sasaki-Einstein

metrics, and balanced metrics. This talk was one of the two which were video-taped at the workshop.

• The second talk which was video-taped was that of *Demailly*. Here a range of important problems around the key notions of volume and higher cohomology analogues for pseudo-effective line bundles was described. Some precise conjectures for these notions in terms of holomorphic Morse inequalities and eigenvalue distribution were formulated.

• An analytic minimal model program, based on the Kähler-Ricci flow with singularities and formulated by *Song* and Tian, was described by *Song*. One particular problem is to determine the Gromov-Hausdorff convergence of the flow.

• An important issue for the theory of complex Monge-Ampère equations is the issue of regularity. Specific questions are: the optimal Hölder regularity for solutions with L^p right-hand sides; the smoothness of solutions, assuming that they are bounded, and satisfy the equation in the sense of pluripotential theory; and the regularity and rank of the Hessian of the solution, when the equation is degenerate.

Soon after the workshop, *Demailly, Dinew and Kołodziej* managed to prove that the Hölder exponent of the solution of the equation with the right hand-side in L^p (this Hölder regularity had been proved by *Kołodziej*) is independent of the geometry of the manifold (it depends only on p and the dimension). More recently, *Székelyhidi and Tosatti* posted a proof of smoothness for a broad class of right hand sides, assuming that the solution is in L^∞. In a different direction, *Błocki* obtained an alternative proof of C^0 estimates for the Calabi-Yau equation for general Hermitian background forms.

• Balanced metrics are a generalization of Kähler metrics which can still provide a deep probe of the complex geometry of the underlying manifold. This appears to be an important direction for further investigations, from both geometric and analytic viewpoints. Many remarkable results and open problems of this nature were described by *Fu* in his talk.

4. List of Participants

Apostolov, Vestislav (Université du Québec à Montréal)
Berman, Robert (Chalmers University)
Błocki, Zbigniew (Jagiellonian University)
Cegrell, Urban (Umeå University)
Czyż, Rafał (Jagiellonian University)
Demailly, Jean-Pierre (Université de Grenoble I)
Dinew, Sławomir (Jagiellonian University)
Edigarian, Armen (Jagiellonian University)
Fu, Jixiang (Fudan University)
Guan, Bo (Ohio State University)
Guan, Pengfei (McGill University)
Guedj, Vincent (Université Aix-Marseille)
Huang, Yong (Chinese Academy of Sciences and McGill University)
Huang, Hongnian (UQAM and McGill University)
Kołodziej, Sławomir (Jagiellonian University)
LaNave, Gabriele (Yeshiva University)
Li, Song-Ying (University of California, Irvine)
Li, Qun (Wright State University)
Lu, Zhiqin (University of California, Irvine)
Phong, Duong (Columbia University)
Pliś, Szymon (Cracow University of Technology)
Song, Jian (Rutgers University)
Spruck, Joel (Johns Hopkins University)
Sturm, Jacob (Rutgers University)
Székelyhidi, Gábor (Columbia University)
Taylor, Al (University of Michigan)
Tosatti, Valentino (Columbia University)
Varolin, Dror (Stony Brook University)
Xu, Lu (Chinese Academy of Sciences and McGill University)
Yau, Shing-Tung (Harvard University)
Yuan, Yuan (Rutgers University)
Zhang, Xiangwen (McGill University)

5. Talks

- Vestislav Apostolov *Extremal Kähler metrics on projective bundles over a curve*
- Robert Berman *Complex Monge-Ampère equations and balanced metrics*
- Jean-Pierre Demailly *Asymptotic cohomology and holomorphic Morse inequalities*
- Sławomir Dinew *Hölder continuity of solutions of Monge-Ampère equations with right hand side in L^p*
- Jixiang Fu *On balanced metrics*
- Bo Guan *Some special Dirichlet problems for the complex Monge-Ampère equation*
- Gabriele La Nave *On the Kähler-Ricci flow and the V-soliton equation*
- Qun Li *Complex Monge-Ampère equations and totally real submanifolds*
- Song-Ying Li *On the rigidity theorems and problems associated to degenerate elliptic operators*
- Zhiqin Lu *Remarks on hypersurface K-stability*
- Jian Song *The Kähler-Ricci flow through singularities*
- Jacob Sturm *Regularity of geodesic rays*
- Gábor Székelyhidi *On convergence of the Kähler-Ricci flow*
- Valentino Tosatti *Complex Monge-Ampère equations on symplectic and Hermitian manifolds*
- Shing-Tung Yau *Canonical metrics and Monge-Ampère equations*