Cross-community collaborations in combinatorics

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1 Overview

In recent years some of the most exciting breakthroughs in Combinatorics on longstanding conjectures have resulted from innovative applications of established techniques to areas where they not necessarily used before. The goal of our workshop was to harness the power of collaboration and bring together open-minded participants with different areas of expertise to produce novel research in a number of globally studied areas. We aspired to create new productive long-term bonds between members of the global community.

A large focus of the workshop was on the training and career enhancement of junior researchers. This was achieved through their fostering new collaborations with world-leading members of the global community during our focused small group work sessions. This gave junior participants opportunities to learn about and work in areas outside of their PhD/postdoctoral focus, gaining invaluable skills and knowledge. They were able to forge meaningful relationships with senior members of the community outside their home institution.

1.1 Workshop objectives

1. A primary objective of the workshop was to stimulate and foster genuinely new (and productive) collaborations amongst participants in topical areas that are not necessarily what they would usually work on and to create *new* long-term bonds between members of the global community.

2. Another key objective of the workshop was the training and career enhancement of junior participants. We have deliberately decided to make the workshop small - 21 people, to not be intimidating for more junior researchers and allow them to flourish. We aspired to a very welcoming and comfortable environment and for them to be able to develop meaningful relationships with senior members of the community.

3. We were committed to ensuring our final participant list is diverse and supports those under-represented in the mathematical sciences. Systematic barriers to inclusion are all too present in our field and we do not wish to enhance the problem.

In the sections below we will detail the scientific progress made during the workshop, and explain how we met each of these objectives.

2 Open Problems

One of the key goals of our workshop was to foster new and exciting collaborations amongst members of the combinatorics community that did not typically work together. We invited all participants to submit well thought out open problems in advance, and begun the workshop with an open problem session where these problems would be presented. In this section we summarise the problems that were suggested for the workshop.

Zero-sum cycles in Abelian groups (suggested by Natasha Morrison)

For a finite abelian group A, define $f(A)$ to be the minimum integer such that for every complete digraph Γ on f vertices and every map $w : E(\Gamma) \to A$, there exists a directed cycle C in Γ such that $\sum_{e\in E(C)} w(e) = 0$. We call w a *weighting* of $E(\Gamma)$ and C a *zero-sum cycle*. The question of determining $f(A)$ arose in a paper by Alon and Krivelevich [**?**] from 2021, who proved that $f(\mathbb{Z}_p) \leq 2p-1$, for p prime, and $f(\mathbb{Z}_q) = O(q \log q)$, for any integer $q \geq 2$. This result was improved upon and generalised by Mészáros and Steiner [**?**], who showed that $f(A) \leq 8|A|$ for any finite abelian group A, and in particular, that $f(\mathbb{Z}_p) \leq \frac{3}{2}p$ for prime p. This was improved to show that $f(B) \leq 2|B|-1$, where B is any finite (not necessarily abelian) group (see [**?**, **?**]. In forthcoming work of Campbell, Hendrey, Gollin, and Steiner, this is improved to a tight bound $f(\mathbb{Z}_q) = q+1$ for every positive integer q.

In [**?**], we show that when p is a prime and $k \geq 1$, then $f(\mathbb{Z}_p^k) \leq 600p \cdot k(\log_2(10k))^2$. We obtain a stronger result when $p = 2$ of $f(\mathbb{Z}_2^k) \leq 600k \log_2(2k)$.

A simple construction shows that $f(\mathbb{Z}_p^k) \geq (p-1)k$. Indeed, let $e_1, \dots, e_k$ be the elementary basis elements of $\mathbb{Z}_p^k$. Consider the complete digraph Γ on $(p-1)k$ vertices, let $\{V_1, \dots, V_k\}$ be an equipartition of $V(\Gamma)$, and label a directed edge xy by e_i whenever $x \in V_i$. It is easy to see that, with this weighting, there are no zero-sum cycles in Γ, as every cycle contains at most $p-1$ edges labelled e_i, for every $i \in [k]$. Thus $f(\mathbb{Z}_p^k) \geq (p-1)k$, as claimed.

Problem 1. *Is it true that $f(\mathbb{Z}_p^k) = O(pk)$?*

Problem 2. *Can improved bounds be found for powers of other Abelian groups? What about $G = \mathbb{Z}_p \times \mathbb{Z}_q$?*

Our proofs in [**?**] rely on bounds on the size of *reduced sets*. Given a multiset S with elements in an abelian group A, the *sumset* of S, denoted $\Sigma(S)$, is defined to be the set of all subset sums of S, namely

$$\Sigma(S) := \left\{ \sum_{t\in T} t : T \subseteq S \right\},$$

where the sum of elements in the empty set is defined to be zero, and hence $|\Sigma(\emptyset)| = 1$. In particular, $0 \in \Sigma(S)$ for every multiset S whose elements lie in A. Say that S is *reduced* if $|\Sigma(S)| > |\Sigma(S')|$, for every S' obtained by removing exactly one element from S. Let $h_p(k)$ be the size of a largest reduced multiset in $\mathbb{Z}_p^k$.

Problem 3. *For p prime, is $h_p(k) = O(pk)$?*

In [**?**] we show that $h_p(k) = O(pk \log(k))$. We have reason to believe that $h_p(2) = 2(p-1)$.

Problem 4. *Let p be prime. Is there a constant C such that $h_p(2) \leq 2p + C$?*

Avoiding progressions in low dimensions (suggested by Gabriel Currier)

Let ℓ_m denote m points on a line with consecutive points of distance 1 apart. In other words, ℓ_m is an m-term arithmetic progression with common difference 1. For finite pointsets S, T, we say that $\mathbb{E}^n \to (S, T)$ if every red/blue-coloring of n-dimensional euclidean space contains either a red (congruent) copy of S or a blue (congruent) copy of T. If there exists a coloring where this is not the case, we say $\mathbb{E}^n \not\to (S,T)$.

For a given s, n, let $t(s,n)$ denote the largest t such that $\mathbb{E}^n \to (\ell_s, \ell_t)$. It was shown, in a recent paper of Conlon and Wu [**?**], that there exists a uniform upper bound on $t(3,n)$ for all $n \geq 2$; in particular, they showed $t(3,n) \leq 10^{50}$. This was subsequently improved to $t(3,n) \leq 1176$ by Führer and Tóth and then $t(3,n) \leq 19$ [**?**] by Currier, Moore and Yip [**?**]. In the other direction, it was shown recently that $t(3,n) \geq 3$ for all $n \geq 2$ [**?**].

To our knowledge, the bounds $3 \leq t(3,n) \leq 19$ are the best known in all dimensions. This is somewhat surprising, since one would expect $t(3,n)$ to grow as n grows. Therefore, we suggest attempting to improve these bounds in lower dimensions.

Problem 5. *Can we show $t(3,2) < 19$? That is, can we construct a red/blue coloring of the plane that avoids red ℓ_3 and blue ℓ_{19}?*

Problem 6. *Similar questions are open for ℓ_4 and ℓ_5. In particular, it is known that $t(4,n) \leq 17$ and $t(5,n) \leq 9$ for all $n \geq 2$ [**?**]. Similarly, can these bounds be improved in low dimensions?*

As mentioned before, the existing colorings are valid in all dimensions. These results rely on showing some equidistribution results about quadratic polynomials mod p, and, somewhat surprisingly, involve almost no geometry. Improving these results in low dimensions would likely have a much more geometric flavor. It is a bit challenging to assess the difficulty of these problems, as (to my knowledge) there are very few known results of this type.

It's an old result of Erdős, Graham, Montgomery, Rothschild, Spencer and Straus [**?**] that $\mathbb{E}^n \not\to (\ell_6, \ell_6)$ for all $n \geq 2$, so the problem of determining $t(6,n)$ reduces to that of determining $t(s,n)$ for $s < 6$. The corresponding problem for ℓ_2 also behaves a bit differently, as it's known there does not exist an upper bound on $t(2,n)$ that is independent of n (in particular, $t(2,n)$ is exponential in n; see, for example, the discussion in [**?**]). However, the upper and lower bounds for $t(2,n)$ in low dimensions are very far apart. In particular, it's known $5 \leq t(2,2) \leq 10^{10}$, with the lower bound due to Tsaturian [**?**], and the upper bound due to Conlon and Fox [**?**].

Problem 7. *Can we get a reasonable upper bound on $t(2,2)$? Say, something on the order of 100?*

A reasonable jumping off point might be [**?**], since this is perhaps the known result that is closest to the above problems. If these are having trouble getting off the ground, there are other low dimensional coloring problems we could consider. For example, it is known, for every 4-point configuration K, that $\mathbb{E}^2 \to (\ell_2, K)$ [**?**]. Also, there exists an 8 point configuration K' such that $\mathbb{E}^2 \not\to (\ell_2, K')$ [**?**]. Thus, we could consider whether there are any smaller configurations for which this holds.

Problem 8. *Does there exist a configuration K of 7 points in the plane such that $\mathbb{E}^2 \not\to (\ell_2, K)$?*

(Edit: it seems this was answered in the positive by Szlam. To beat the best known result we would need to find a configuration of 6 points.

Rainbow path coverings (suggested by Bertille Granet)

Answering an old conjecture of Chung [**?**], Fan [**?**] showed that the edges of any connected graph on n vertices can be covered by at most $\lceil \frac{n}{2} \rceil$ (not necessarily edge-disjoint) paths. One can see that this result is tight by considering complete graphs: since any path in K_n covers at most $n-1$ edges, one needs at least $\binom{n}{2}(n-1)^{-1} = \frac{n}{2}$ paths to cover $E(K_n)$.

An edge-coloured graph is called *rainbow* if all of its edges have distinct colours. Bonamy, Botler, Dross, Naia, and Skokan [**?**] recently asked for a rainbow analogue of the above mentioned path covering result.

Problem 9 ([**?**])**.** *Does every properly edge-coloured graph G contain $O(|V(G)|)$ rainbow paths that cover $E(G)$?*

In [**?**], the authors claimed that they can verify the weakening of Problem 9 where "paths" is replaced by "trails". Additionally, Kaique, Mota, and Naia [**?**] considered Problem 9 for random graphs. More precisely, they showed that if $p = \omega((\frac{\ln n}{n})^{\frac{1}{2}})$ and $G \sim G(n,p)$, then with high probability every proper edge-colouring of G contains $O(|V(G)|)$ rainbow paths that cover $E(G)$. They also observed that for $p = O(n^{-1})$, then with high probability $|E(G)| = O(n)$ and so the edges of any proper edge-colouring of G can be trivially decomposed into $O(n)$ rainbow paths.

If Problem 9 proves to be too challenging in general, then a natural stepping stone would be to consider specific classes of graphs, such as dense graphs or $G(n,p)$ in the missing range of p for example.

Problem 10 ([**?**])**.** *Let $\Omega(n^{-1}) \leq p \leq O((\frac{\ln n}{n})^{\frac{1}{2}})$ and $G \sim G(n,p)$. Show that with high probability every proper edge-colouring of G contains $O(|V(G)|)$ rainbow paths that cover $E(G)$.*

Ramsey number of linear hypergraph trees (suggested by Matías Pavez-Signé)

Let $k \geq 2$ and $1 \leq \ell \leq k-1$. We say that a k-uniform hypergraph T is an ℓ-tree if there is an ordering $e_1, \ldots, e_n$ of $E(T)$ such that for each $2 \leq i \leq n$ there exists an index $1 \leq j \leq i-1$ so that

T1 $|e_i \cap e_j| = \ell$, and

T2 $\left|e_i \setminus \bigcup_{1\leq s\leq i-1} e_s\right| = k - \ell$.

Condition **T1** says that whenever we add a new edge it must intersect in exactly ℓ vertices with a previous edge, and condition **T2** says that we add exactly $k-\ell$ new vertices at each step. The case $\ell = k-1$ is referred to as *tight trees* and the case $\ell = 1$ as *loose trees*.

I'm interested in trying to estimate the 2-colour Ramsey number of an arbitrary loose tree T with n edges. In the case of graphs ($k = 2$), Burr and Erdős conjectured in the 70s that for every tree T with n edges, $R(T) \leq 2n$, which was confirmed for large n by Yi Zhao in 2008 using the regularity method. This bound is sharp when T is a star, though is not tight in general (there is another conjecture by Burr from 1974 which gives better estimates depending on the tree, but remains open).

For an arbitrary loose tree, I don't expect to find exact results yet but rather find a general upper bound as in the Burr–Erdős conjecture. The only known result that I'm aware of is the case of loose paths (which is one of the canonical examples of hypergraph loose trees). For $k \geq 2$ and $n \geq 1$, let $P_{1,n}^{(k)}$ denote the k-uniform loose path with n edges, that is, the k-graph with vertices $v_1, \ldots, v_{n(k-1)+1}$ and edges $e_1, \ldots, e_n$ such that, for $i \in [n]$,

$$e_i = \{v_{(i-1)(k-1)+1}, \ldots, v_{(i-1)(k-1)+k}\}.$$

For loose paths, we have an asymptotic result working for any uniformity k and an exact result which works for uniformity $k \geq 8$.

- Gyárfás–Sárközy–Szemerédi (2008): $R(P_{1,n}^{(k)}) = (2k-1)\frac{n}{2} + o(n)$.
- Omidi–Shahsiah (2017): If $k \geq 8$ and $n \geq 3$, $R(P_{1,n}^{(k)}) = (k-1)n + \lfloor \frac{n+1}{2} \rfloor$.

The lower bound comes from the following canonical construction. Consider two disjoint red cliques with vertex sets A and B, respectively, such that $|A| = (k-1)n$ and $|B| = \lfloor \frac{n-1}{2} \rfloor$. Between A and B colour every possible edge in blue. It is clear that $P_{1,n}^{(k)}$ does not embed in red as $|A|, |B| \leq (k-1)n < |V(P_{1,n}^{(k)})|$. If we try to embed $P_{1,n}^{(k)}$ in blue, the best strategy is to embed $\{v_{(2i-1)(k-1)+1}\}_{i \leq \lceil \frac{n}{2} \rceil}$ into B and then connect through A. This is not possible though as $|B| < \lceil \frac{n}{2} \rceil$.

In particular, this result implies that any general bound that could hope to find should be at least $(k-1)n + \lfloor \frac{n+1}{2} \rfloor$. In the case of graphs, a path gives the lowest possible Ramsey number amongst all trees with a fixed number of edges and so it's natural to expect the same thing happening for hypergraphs. Although I couldn't find any better colouring for uniformity $k \geq 3$, I tend to think that the upper bound should be $kn + o(n)$ rather than $(2k-1)\frac{n}{2} + o(n)$.

Problem 11. *Let T be a k-uniform loose tree with n edges. Is it true that $R(T) \leq kn + o(n)$?*

An intermediate step would be considering a special case of loose trees called k-expansion trees. Given a 2-uniform tree T with n edges, the k-expansion of T is the k-uniform graph obtained from T by adding $(k-2)$ new vertices to each edge. We say that a k-graph T is a k-expansion tree if there is a 2-uniform tree T' such that T is the k-expansion of T'.

Problem 12. *Let T be a k-expansion tree with n edges. Is it true that $R(T) \leq kn + o(n)$?*

Induced cycles (suggested by Sophie Spirkl)

The first question is due to Linda Cook:

Problem 13. *Given p and q, is there a polynomial-time algorithm for deciding if a given graph G contains an induced cycle of length congruent to p mod q? What if I also want it to have length at least some ℓ?*

This is a problem with p, q, ℓ fixed; so the running time can depend on these in arbitrary ways. The answer is known for any p, ℓ when $q = 2$ [**?**, **?**].

There's particular interest in the case $p = 0$ and $q = 3$ (see `https://gilkalai.wordpress.com/2014/12/19/when-a-few-colors-suffice/`); [**?**] showed that these graphs have bounded chromatic number (later generalized by [**?**]). I believe it's conjectured that they are always 3-colourable?

Problem 14. *Are graphs with no induced cycles of length 0 mod 3 always 3-colourable?*

I'm not sure whose question this is; I believe I heard it from Marthe Bonamy.

I actually don't know the answer to the following question, either (but it might be known to other people):

Problem 15. *What happens in Problem 13 if we drop the word "induced"?*

Incidentally, Problem 14 is known to be true if we drop the word "induced" [**?**].

At some point, there was a conjecture by Dan Král' that maybe every graph as in Problem 14 has an edge you can delete and preserve the property in question; but this was disproved by Marcin Wrochna.

Multiplicative graphs (suggested by Anna Gujgiczer)

A graph K is called multiplicative if $G \times H \to K$ implies $G \to K$ or $H \to K$ for every graph pair G and H (where $\to$ denotes the existence of graph homomorphism). Hedetniemi conjectured in 1966 that every complete graph is multiplicative. This conjecture is now refuted in general, but motivated by this question multiplicaticity of other graphs were studied as well. Some graph operations proved to be useful in this type of questions:

Definition 16 (Γ_k). *For any odd k*
$V(\Gamma_k(G)) = V(G)$,
$E(\Gamma_k(G)) = \{\{u, v\} | u$ *and v are endpoints of a walk of length exactly k in G*$\}$.

Definition 17 (Ω_k). *For any odd $k = 2s - 1$*
$V(\Omega_k(G)) = \{(A_0, , \ldots, A_{s-1}) : \forall i\ A_i \subseteq [t], |A_0| = 1, |A_1| \geq 1, \forall i\ A_i \subseteq A_{i+2}, A_{s-2} \bowtie A_{s-1}\}$,
$E(\Omega_k(G)) = \{(A_0, \ldots, A_{s-1}), (B_0, \ldots, B_{s-1})\} :\ \forall i\ A_{i-1} \subseteq B_i, B_{i-1} \subseteq A_i$ *and* $A_{s-1} \bowtie B_{s-1}\}$,
where $A \bowtie B$ means, that every vertex of A is connected to every vertex of B.

These two graph operations are left and right adjoints, meaning, that we have $\Gamma_k(G) \to H$ if and only if $G \to \Omega_k(H)$. In [**?**] it is shown that K is multiplicative if and only if $\Omega_3(K)$ is multiplicative (and this result generalizes easily for any odd k). Moreover, we always have that $\Gamma_k(\Omega_k(K))$ is homomorphically equivalent to K. The interesting case is when $\Omega_k(\Gamma_k(K))$ is homomorphically equivalent to K as well. This property can be used to prove that $\Gamma_k(K)$ is multiplicative, given that K is. (If K is multiplicative then $\Omega_k(\Gamma_k(K))$ is multiplicative, if $\Omega_k(\Gamma_k(K))$ is multiplicative then $\Gamma_k(K)$ is as well.)

In [**?**] they proved that if K is a graph with girth at least 13, then $\Omega_3(\Gamma_3(K))$ is homomorphically equivalent to K. Using this and the fact, that square-free graphs are multiplicative [**?**] we get, that if K has girth at least 13, then $\Gamma_3(K)$ is multiplicative (which graph class was different from the previously known multiplivative graphs). The authors conjecture, that we can get a similar statement for other odd k values as well.

Problem 18. *Is it true, that if K has girth at least $4k+1$, then $\Omega_k(\Gamma_k(K))$ is homomorphically equivalent to K?*

In general, it would be interesting to see when we can have this property.

Problem 19. *For which graphs do we have that $\Omega_k(\Gamma_k(K))$ is homomorphically equivalent to K?*

Another direction would be to investigate a "relaxed" version of multiplicativity. In [**?**] Wrochna introduced a relaxed version of Hedetniemi's conjecture (which would have interesting topological implications). It is the following: for every positive integer n and graph pair G and H, $G \times H \to K_n$ implies $\Omega_k(G) \to K_n$ or $\Omega_k(H) \to K_n$ (for some large enough odd k). In my knowledge this question is not well investigated. We can change K_n to any known non-multiplicative graph and ask a similar question.

Problem 20. *For which known non-multiplicative graphs K do we have relaxed multiplicativity i.e. that $G \times H \to K$ implies $\Omega_k(G) \to K$ or $\Omega_k(H) \to K$ (for some large enough odd k)? The known non-multiplicatve graphs so far:*

- *$K = G \times H$, s.t. $G \nrightarrow H$ and $H \nrightarrow G$,*
- *some Kneser graphs (see [**?**]),*
- *K_n for $n \geq 4$.*

Discrepancy (suggested by Joseph Hyde and Amedeo Sgueglia)

In discrepancy theory, the basic question is whether a structure can be partitioned in a balanced way, or if there is always some discrepancy no matter how the partition is made. Questions of this type were first considered by Erdős in the 60s (see [**?**, **?**]) and we propose several discrepancy problems in different contexts.

More formally, for a set Ω and a family $\mathcal{A}$ of subsets of Ω, we define the *discrepancy* of $\mathcal{A}$ as

$$\min_c \max_{A \in \mathcal{A}} \left| \sum_{x \in A} c(x) \right|$$

where the minimum runs over all colourings $c : \Omega \to \{-1, 1\}$.

General discrepancy

Gishboliner, Glock and Sgueglia [**?**] suggested to investigate how robust is discrepancy, possibly pursuing the following line of research.

Problem 21 ([**?**])**.** *Consider a random subset $\Omega' \subseteq \Omega$ obtained by including each element of Ω independently with probability $1/2$, and let $\mathcal{A}' \subseteq \mathcal{A}$ be the family of surviving sets. Can we (non-trivially) lower bound the discrepancy of $\mathcal{A}'$ in terms of the discrepancy of $\mathcal{A}$?*

Tight Hamilton cycles

In the context of (hyper)graphs, Ω is the edge set of some graph and $\mathcal{A}$ corresponds to a family of sub(hyper)graphs. A popular recent area of research has been to determine how large the minimum degree of an n-vertex hypergraph H needs to (asymptotically) be so that every 2-colouring of the edges of H contains a spanning subgraph of a certain type with discrepancy $\Omega(n)$ (we will say that such a subgraph has *high discrepancy* in this context).

This type of problem has been considered for graphs [see, e.g., [**?**, **?**]], but not for hypergraphs until very recently. For every $\varepsilon > 0$, Gishboliner, Glock and Sgueglia [**?**] showed that if n is sufficiently large, every n-vertex k-uniform hypergraph H with $\delta_{k-1}(H) \geq (1/2 + \varepsilon)n$ has a Hamilton cycle with high discrepancy. This is a discrepancy version of a famous result of Rödl, Ruciński and Szemerédi [**?**], and is asymptotically best possible as there are k-uniform hypergraphs H with $\delta_{k-1}(H) = n/2 - O(1)$ which do not have a tight Hamilton cycle (or even a perfect matching).

Problem 22 ([**?**]). *For $1 \leq \ell \leq k-2$, which ℓ-degree forces the existence of a tight Hamilton cycle with high discrepancy?*

While for $\ell = k-1$ the existence threshold is asymptotically the same as the discrepancy threshold, this is not always the case. For example, Reiher, Rödl, Ruciński, Schacht and Szemerédi [**?**] showed that for n sufficiently large an n-vertex 3-graph H with $\delta_1(H) \geq (5/9 + o(1))\binom{n}{2}$ has a tight Hamilton cycle. However, Balogh, Trelown and Zárate-Guerén [**?**] constructed the following 3-graph H on n vertices, with n divisible by 2 and $\delta_1(H) \geq \frac{3}{4}\binom{n-1}{2}$ such that every tight Hamilton cycle in H has precisely $n/2$ edges in each colour: Partition $V(H)$ into two vertex sets A, B with $|A| = |B| = n/2$ and take all 3-edges containing at least one vertex from each of A and B. Then colour each edge with two vertices in A with red and each edge with two vertices in B with blue.

We remark that we are aware of a group of people trying to show that $\delta_1(H) \geq (3/4 + \varepsilon)\binom{n-1}{2}$ forces a perfect matching of high discrepancy. Moreover, we remark that [**?**] showed that for $2 \leq \ell \leq k-1$, the existence and the discrepancy thresholds for a perfect matching are asymptotically the same. It is therefore natural to investigate if this too is the case for tight Hamilton cycles.

We remark that, while for conciseness we only introduced discrepancy for 2 colours, all the results stated above have been proved for any number of colours.

Oriented discrepancy problems

A different take on discrepancy was introduced by Gishboliner, Krivelevich and Michaeli in [**?**] for oriented graphs. For an oriented graph G and $x, y \in V(G)$, we write xy to mean an edge oriented from x to y. Say $C = v_1 \dots v_\ell v_1$ is a cycle in an oriented graph G. Let $\sigma^+(C)$ and $\sigma^-(C)$ denote the number of forward and backward edges in C, respectively; that is, $\sigma^+(C) := |\{1 \leq i \leq \ell : v_i v_{i+1} \in E(G)\}|$ and $\sigma^-(C) := |\{1 \leq i \leq \ell : v_{i+1} v_i \in E(G)\}|$, with indices taken modulo ℓ. Further, let $\sigma_{\max}(C) := \max\{\sigma^+(C), \sigma^-(C)\}$ and $\sigma_{\min}(C) := \min\{\sigma^+(C), \sigma^-(C)\}$. Note that $\sigma_{\max}(C)$ and $\sigma_{\min}(C)$ both remain the same irrespective of the 'direction' the labelling of the vertices of C is produced in.

Freschi and Lo [**?**] recently proved the following, resolving a conjecture from [**?**]. (Gishboliner, Krivelevich and Michaeli [**?**] previously proved an approximate version of their conjecture.)

Theorem 23. *Let G be an oriented graph on $n \geq 3$ vertices with $\delta(G) \geq n/2$. Then there exists a Hamilton cycle C in G such that $\sigma^{\max}(C) \geq \delta(G)$.*

Freschi and Lo [**?**] suggest a number of possible directions.

Problem 24 ([**?**]). *Prove an Ore-type[1] version of Theorem 23.*

Problem 25 ([**?**]). *Consider other structures with a natural concept of 'direction' built into them, such as powers of Hamilton cycles.*

They also suggest the following land to explore.

Problem 26 ([**?**]). *Given a graph H, one can assign an arbitrary orientation to H. We then say that a copy of H in G has large oriented discrepancy if significantly more than half of its edges agree (or disagree) with the initial orientation. It would be interesting to determine which edge-orientation of H minimises (or maximises) the minimum degree threshold required to ensure a host graph G contains a copy of H with a certain amount of discrepancy.*

Restricted solutions to linear equations and the number of Hadamard matrices (Suggested by Asaf Ferber)

Many problems in combinatorial linear algebra require upper bounds on the number of solutions to an underdetermined system of linear equations $Ax = b$, where the coordinates of the vector x are restricted to take

[1] An 'Ore-type' condition is usually a bound on the sum of degrees of non-adjacent pairs of vertices in a graph. For example, Ore's theorem is the following: Let $n \geq 3$. For an n-vertex graph G, if $d_G(x) + d_G(y) \geq n$ for all non-adjacent pairs of vertices $x, y \in V(G)$, then G contains a Hamilton cycle.

values in some small subset (e.g., $\{\pm 1\}$) of the underlying field. More specifically, let $a_1, \ldots, a_m \in \mathbb{R}^n$, and let $x_1, \ldots, x_m$ be i.i.d random variables taking values in some given set $X \subseteq \mathbb{R}$. We wish to study the following quantity: $\sup_{b \in \mathbb{R}^n} \Pr\left(\sum_{i=1}^m a_i x_i = b\right)$.

An initial motivation for me to study the above quantity came from the following problem: A square matrix H of order n whose entries are in $\{\pm 1\}$ is called a *Hadamard matrix of order* n if its rows are pairwise orthogonal (i.e. if $HH^T = nI_n$). Hadamard matrices are named after Jacques Hadamard, who studied them in connection with his maximal determinant problem. Specifically, Hadamard asked for the maximum value of the determinant of any $n \times n$ square matrix, all of whose entries are bounded in absolute value by 1. He proved that the value of the determinant of such matrices cannot exceed $n^{n/2}$. Moreover, he showed that Hadamard matrices are the only ones that can attain this bound. Due to this fact, Hadamard matrices have been the focus of considerable attention from many different communities: coding theory, design theory, statistical inference, and signal processing, to name a few.

Hadamard matrices of order 1 and 2 are trivial to construct, and it is quite easy to see, by considering the first three rows, that every other Hadamard matrix (if it exists) must be of order $4m$ for some $m \in \mathbb{N}$. Whereas Hadamard matrices of infinitely many orders have been constructed, the question of whether one of order $4m$ exists for every $m \in \mathbb{N}$ is the most important open question on this topic, and remains wide open.

Conjecture 27 (The Hadamard conjecture)**.** *There exists a Hadamard matrix of order* $4m$ *for every* $m \in \mathbb{N}$.

As explicit constructions are very hard to obtain, we tried to show that there are not "too many" Hadamard matrices of a given order $4m$. To see that connection to vector anti-concentration inequalities, imagine that one builds a Hadamard matrix row by row. Let M_k be the partial $k \times n$ matrix comprising the first k rows. Clearly, to add the $(k+1)$st row, one needs to find a vector $x \in \{\pm 1\}^n$ which is orthogonal to the row-space of M_k. In other words, one wants to find such an x with $M_k x = 0$. Now, to obtain an upper bound on the number of Hadamard matrices, one needs to upper bound the number of solutions to such linear system under the restriction that $x \in \{\pm 1\}^n$. Together with Jain and Zhao [**?**] we showed that there are at most $2^{(1/2-\epsilon)n^2}$ Hadamard matrices of order n, which is the first non trivial bound known. Moreover, we made the following conjecture:

Conjecture 28. *There are* $2^{O(n \log n)}$ *Hadamard matrices of order* n.

I would be very interested in getting any bound of the form $2^{n^{1+o(1)}}$.

Another related problem is the following problem regarding the intersection of a random vector space with the hypercube. Let $v_1, \ldots, v_m \in \{\pm 1\}^n$ and let V_m be their span. Clearly, as $\pm v_i \in \{\pm 1\}^n$ for all i, it follows that

$$|V_m \cap \{\pm 1\}^n| \geq 2n.$$

We say that V_m *has a non-trivial intersection with the hypercube* if the inequality above is strict.

A classical result of Odlyzko [**?**] from 1986 asserts that, for randomly chosen $v_1, \ldots, v_m \in \{\pm 1\}^n$, the probability that V_m has a non-trivial intersection with the hypercube is at most $4\binom{m}{3}\left(\frac{3}{4}\right)^n + O\big(.7^n\big)$, as long as $m \leq n - 10n/\log n$. The main term in the estimate is the best possible, as it is the probability that some three of these vectors have another ± 1 vector in their span. Many years later, in their breakthrough result about the singularity problem, Kahn, Komlós, and Szemerédi [**?**] have managed to extend Odlyzko's result to $m \leq n - C$, where C is some large constant (same main term on the probability).

Observe that if we could take $C = 1$ in the above result, then one immediately obtains an upper bound of around $4\binom{n}{3}\left(\frac{3}{4}\right)^n$ on the probability that a uniformly random $n \times n$ matrix with ± 1 entries is singular: Expose the first $n-1$ rows. With probability at most $4\binom{n-1}{3}\left(\frac{3}{4}\right)^n$ their span has a non trivial intersection with the hypercube. Condition on the intersection being trivial, the probability that the last row belongs to the span of the first $n-1$ rows is $2n2^{-n}$.

Here, we are interested in a *large deviations* version of this problem. Namely:

Problem 29. *Let* $m \leq n$*, and let* $v_1, \ldots, v_m \in \{\pm 1\}^n$ *randomly chosen (independently of each other). Estimate, for a given* $t > 2n$*, the quantity*

$$Q(t, n) = \Pr\left(\left|V_m \cap \{\pm 1\}^n\right| \geq t\right).$$

I believe that even a partial solution to Problem 29 will have far-reaching applications. My favorite regime to begin with is $t \geq 2^{\varepsilon n}$, and in this case I believe that $Q(n, t) \leq 2^{-\Theta(n^2)}$.

Covering grids with multiplicity (suggested by Shagnik Das)

It is easy to see that you can cover all the points of $\{0,1\}^d$ with two affine hyperplanes, and that this is best possible. However, a famous theorem of Alon and Füredi [**?**] shows that if you must avoid the origin, then it takes at least d hyperplanes to cover all other points of the hypercube.

Clifton and Huang [**?**] recently revived interest in this problem by considering a multiplicity version of this problem — let $f(d,k)$ denote the minimum number of hyperplanes needed to cover all nonzero points of $\{0,1\}^d$ at least k times, while avoiding the origin entirely. They used linear programming methods to show $f(d,k) = (H_d + o(1))k$ for fixed d and large k, where H_d is the dth Harmonic number. In the other regime, where k is fixed and d is large, they extended the algebraic approach of Alon and Füredi to establish a lower bound of $f(d,k) \geq d+k+1$ for $k \geq 4$ and $d \geq 3$.

Soon afterwards, Sauermann and Wigderson [**?**] solved the algebraic version of the problem — they proved that for $k \geq 3$ and $d \geq 2k-3$, any polynomial in $P \in \mathbb{R}[x_1, x_2, \ldots, x_d]$ with $P(\vec{0}) \neq 0$ that has zeroes of multiplicity at least k at all other points of $\{0,1\}^d$ must have degree at least $d+2k-3$, and that this bound is best possible.

This immediately improves the lower bound for the Clifton–Huang problem to $f(d,k) \geq d+2k-3$ when $d \geq 2k-3$. However, Clifton and Huang conjecture that more hyperplanes are needed. An ambitious/longer-term goal would be to prove them right.

Problem 30. *Show that for every fixed k, when d is sufficiently large, we have $f(d,k) \geq d + \binom{k}{2}$.*

If true, this would be best possible — take the standard hyperplanes $H_i = \{\vec{x} : x_i = 1\}$ for $i \in [d]$, together with, for $1 \leq s \leq k-1$, $k-s$ copies of the hyperplane $\{\vec{x} | \sum_i x_i = s\}$. Some evidence in favour of the conjecture was provided by Alon, who used hypergraph Ramsey to prove that when d is very large, if we assume that we have the d standard hyperplanes $\{H_i : i \in [d]\}$, then we do need at least $\binom{k}{2}$ additional hyperplanes to complete the cover.

This problem might be challenging, since the Sauermann–Wigderson result suggests that the pure algebraic approach will not be enough, and instead a combination of algebraic and geometric or combinatorial arguments may be needed. For the purposes of this workshop, I propose working in a related but seemingly simpler setting.

Given sets $S_1, S_2 \subset \mathbb{R}$, each containing 0 and of size n, consider the two-dimensional grid $\Gamma = S_1 \times S_2$. We define a k-cover of Γ to be a set of lines that avoids the origin and covers all other points of Γ at least k times, and denote by $\mathrm{cov}_k(\Gamma)$ the minimum size of a k-cover of Γ.

A general theorem of Ball and Serra [**?**], which extends the algebraic arguments of Alon and Füredi, shows that we have $\mathrm{cov}_k(\Gamma) \geq (k+1)(n-1)$, and this is tight when $k=1$. However, together with Bishnoi, Boyadzhiyska and den Bakker [**?**], we showed that this is generally not tight for $k \geq 2$. More precisely, we used the linear programming approach of Clifton and Huang to establish the general bounds $(10 - 4\sqrt{5} + o(1))k(n-1) \leq \mathrm{cov}_k(\Gamma) \leq \lceil \frac{3k}{2} \rceil (n-1)$, where the asymptotics are as $n \to \infty$.

Moreover, we show that whenever Γ is such that any non-horizontal and non-vertical line contains at most $o(n)$ points of Γ, then $\mathrm{cov}_k(\Gamma) \geq (\frac{3}{2} + o(1))k(n-1)$. In particular, this shows the upper bound is close to tight for almost all grids Γ.

However, the most natural grid, $\Gamma_n = \{0,1,\ldots,n-1\} \times \{0,1,\ldots,n-1\}$, does not satisfy this condition. For this special case, we prove (as $k, n \to \infty$)

$$(2 - e^{-1/2} + o(1))k(n-1) \leq \mathrm{cov}_k(\Gamma_n) \leq (\sqrt{2} + o(1))k(n-1).$$

We prove the upper bound by explicit construction of a k-cover of Γ_n that only uses lines of slope 0, ∞ and -1, and believe that it gives the correct value of $\mathrm{cov}_k(\Gamma_n)$. To support our beliefs, we further prove that any cover using only these three types of lines must have size at least $(\sqrt{2} + o(1))k(n-1)$. I think it is not beyond the realms of possibility to prove this unconditionally.

Problem 31. *For the standard grid $\Gamma_n = \{0,1,\ldots,n-1\} \times \{0,1,\ldots,n-1\}$, show that, as $k, n \to \infty$, any set of lines in $\mathbb{R}^2$ that avoids the origin and covers all other points of Γ_n at least k times must have size at least $(\sqrt{2} + o(1))k(n-1)$.*

Our proofs suggest that the difficulties arise from lines of slope 1 — these can contain many points of Γ (and thus be efficient in a k-cover), but do not seem to fit together well with the lines of slope -1. Thus, it may be helpful as a first step to establish the lower bound for covers with the following restrictions:

(i) only contain lines of slope $0, \infty, -1$ or 1, or

(ii) do not contain lines of slope 1.

Finally, independently of determining the explicit value of $\mathrm{cov}_k(\Gamma_n)$, we suspect the standard grid might be the easiest to cover, and it would be interesting to see if this can be proven.

Problem 32. *Let $\Gamma \subset \mathbb{R}^2$ be an $n \times n$ grid. Is it true that we have $\mathrm{cov}_k(\Gamma) \geq \mathrm{cov}_k(\Gamma_n)$?*

Random inner-product sets (suggested by Aditya Potukuchi)

Let q be an odd prime power (the odd assumption is convenient but AFAIK not necessary). For subsets $A, B \subseteq \mathbb{F}_q^2$ let us define their inner product set

$$\langle A, B\rangle = \{\langle a, b\rangle \mid a \in A,\ b \in B\}.$$

In general $\langle A, B\rangle$ could have much smaller size than A or B. For instance if $A = B = \{\alpha x : \alpha \in \mathbb{F}_q,\ x \in \mathbb{F}_q^2\}$, we have that $|\langle A, B\rangle| \leq (|A| + 1)/2$. The main intuition behind the following conjecture is that this isn't typically the case, and that for a small randomly chosen B, every $\langle A, B\rangle$ should be large

Conjecture 33. *There is a large enough constant C and a small enough constant ϵ such that for a uniformly randomly chosen set $B \subseteq \mathbb{F}_q^2$ of size $C \log q$, w.h.p.*

$$\min_{\substack{A \subseteq \mathbb{F}_q^2 \\ |A| \leq \epsilon q}} |\langle A, B\rangle| \geq |A|/2.$$

Remark: *It's not a bad idea (so far) to replace 'inner-product' with any symmetric bilinear function that you might find more natural, for example $\mathrm{Tr}(\alpha_x \cdot \alpha_y)$, for a linear $\alpha : \mathbb{F}_q^2 \to \mathbb{F}_{q^2}$. My original motivation comes from a different function, but it seems there's a nice (in my opinion) interpretation of the same context for many different choices of the function.*

This problem originally comes from the complexity theoretic side of coding theory (even the case $d = 1$ is interesting for what follows): The A-punctured degree-d Reed-Solomon code is the set

$$RS_d(A) := \{(f(x))_{x \in A} \mid f \in \mathbb{F}_q[X],\ \deg(f) \leq d\},$$

i.e., the evaluation sets of all polynomials of degree at most d over $\mathbb{F}_q$. Here, I am looking at polynomials as functions from $\mathbb{F}_q$ to $\mathbb{F}_q$. This has several properties that make it very interesting to study as vector spaces, as graphs, as codes, etc.. One interesting property is that of *list-recoverability*. We say that $RS_d(A)$ is (ℓ, L)-list recoverable if for every family of subsets $\{S_\alpha\}_{\alpha \in A} \subset \mathbb{F}_q$, each of size at most ℓ, it holds that

$$|RS_d(A) \cap (\times_{\alpha \in A} S_i)| \leq L.$$

The following is known to be an old problem in this field, but I can't find it written down explicitly anywhere. Venkat Guruswami told me that he was one of the people who (independently) started thinking about this at the same time:

Problem 34. *For $A \subseteq \mathbb{F}_q$ randomly chosen of size $\Omega_d(\log q)$, is it true that $RS_d(A)$ is $(\Omega_d(q), O_d(q))$-list recoverable?*

The most relevant progress towards this problem as stated in my (maybe biased) opinion is [**?**], which proves the statement for $|A| \geq \Omega_d(\sqrt{q} \log q)$. There have been some papers since that reduce the size of A by sub-polynomial factors, but with *much* smaller ℓ, L compared to q. Conjecture 33 would prove the above statement, but for $(\Omega_d(q), O_d(q \log q))$-list recoverablility, which is not only much more manageable, but probably not far off from answering the full question (assuming it's true).

3 Presentations

In this section we give details on the talks at the workshop. We invited a small number of senior researchers to give talks on powerful current methods or exciting recent results. Towards are goal of training younger researchers, we encouraged any non-senior researcher who wanted to speak to volunteer give a talk and had presentations from Stacie Baumann, Anna Gujgiczer, Aditya Potukuchi, and Amedeo Sgueglia.

3.1 Plenary talks

Speaker: Shagnik Das
Title: Explicit constructions of strong blocking sets and minimal codes
Abstract: A strong blocking set in a finite projective space is a set of points that intersects each hyperplane in a spanning set. In this talk we present a new graph theoretic construction of such sets: combining constant-degree expanders with asymptotically good codes, we explicitly construct strong blocking sets in the $(k-1)$-dimensional projective space over F_q that have size $O(qk)$. Since strong blocking sets have recently been shown to be equivalent to minimal linear codes, our construction gives the first explicit construction of F_q-linear minimal codes of length n and dimension k, for every prime power q, for which $n = O(qk)$. This is joint work with Noga Alon, Anurag Bishnoi and Alessandro Neri.

Speaker: Asaf Ferber
Title: Quantum algorithms on graphs
Abstract: In this talk I'll give a short introduction to quantum computation and will illustrate how to utilize it in order to speed up some classical graph algorithms. In particular, we will present an asymptotically tight result for learning a Hamilton cycle using OR querries, and obtain a polynomial improvement for the best known $(\Delta + 1)$-coloring graph quantum algorithm for coloring a graph with maximum degree Delta. This is based on joint works with Liam Hardiman (UCI), and with Xiaonan Chen (UCI) and Liam Hardiman.

Speaker: Sophie Spirkl
Title: Induced subgraphs and treewidth
Abstract: Treewidth is a graph parameter which is useful both for structure and for algorithms. Robertson and Seymour showed which graph minors and which subgraphs "cause" large treewidth. However, the question "Which induced subgraphs cause large treewidth?" is still wide open, and I'll you some pieces of the answer that have been found so far.

Speaker: Bhargav Narayanan
Title: Anticoncentration and Antichain Codes
Abstract: A basic problem at the intersection of probability and combinatorics is the Littlewood-Offord anti-concentration problem: given real numbers $a_1, \ldots, a_n$, what is the largest possible point probability of the random sum $a_1X_1 + \ldots + a_nX_n$ for iid Bernoulli random variables $X_1, \ldots, X_n$? Several variants of this problem, involving additional arithmetic constraints on the numbers $a_1, \ldots, a_n$, have proved to both be deep and widely applicable; two notable examples of such variants include the Sarkozy-Szemeredi theorem (resolving the Erdos-Moser problem) and Halasz's theorem. A few years ago, it became evident to me that all these arithmetic results are in fact specializations of a more abstract, purely combinatorial phenomenon. In this talk, I will take the scenic route to the recent proof (with Ben Gunby, Xiaoyu He and Sam Spiro) of such an abstract result, regarding the density of "antichain codes in the Boolean hypercube, surveying the history of these problems and some of the many applications along the way.

Speaker: Julia Böttcher
Title: Graph universality
Abstract: Given a class G of n-vertex graphs, how can we construct a host graph H that contains them all as subgraphs? Graphs H with this property are called universal for G, and the question gets interesting when we put certain restrictions on H. For example, we might be interested in a graph H with as few edges as possible, or a graph H which has only n vertices itself and still only few edges. Or we might ask when certain random graphs are universal for G. This all leads to a variety of interesting and challenging problems. In the talk, I

will explain what is known and what is open for some classes of graphs G. I will also detail some techniques that I recently used with my co-authors Peter Allen and Anita Liebenau for progress when G consists of all D-degenerate graphs for a fixed D.

4 Training and career enhancement of junior participants

The workshop was carefully designed to maximise the benefit to less senior researchers (details of how we did this can be seen in our original proposal). As organisers, we designed the working groups in such a way as to ensure that all junior researchers were in some group with more senior participants, so that they could network and learn from their expertise. We also planned group social activities in the evenings to facilitate networking in a more relaxed setting. We believe we were very successful in these goals, as we have received several very positive emails from participants after the workshop thanking us for the invitation and telling us what they gained from the experience.

One PhD student wrote: *"The workshop was a great opportunity, and although we don't have any exciting progress to report at the moment, we are planning to keep working on our project. Overall it has been a very friendly and stimulating experience, and I have met many new people. Hopefully this will lead to new collaborations and ideas in the future! Many thanks!"*

Another PhD student wrote: *"What a lovely conference! As an early-career researcher, CCCIC was extremely helpful in a variety of ways. First, it provided a relaxed and friendly environment in which I could develop relationships both with my peers and with more senior researchers. Second, I was able to learn lots of new math, both from talks and group work, and (maybe even more importantly) a number of excellent open problems that I'm continuing to think about. Finally, on a more concrete level, I left BIRS with a planned collaboration, and an invitation to another workshop! I've not had experiences like this at other conferences, and I'm quite sure these opportunities resulted directly from the close-knit environment of the workshop. I'm very grateful to have had the opportunity!"*

5 Equity, Diversity and Inclusion

In order to achieve our third objective, we spent a lot of time and consideration before the workshop on ensuring our final participant list was diverse and strongly includes those from groups under-represented in the mathematical sciences.

More details about our nomination process can be found in our EDI statement, we present a summary here. We solicited nominations for PhD students and postdocs to invite, both from invited participants and other members of the community. This resulted in us being able to invite many wonderful young researchers that were not already personally known to us the organisers. This resulted in all participants meeting new colleagues and future collaborators at the workshop. We asked in particular for nominations of those under-represented in the mathematical sciences and for nominations of participants that were based at different institutions to the nominator (to try to ensure diversity and minimise nepotism).

As stated in our proposal, we wrote that *"We are aiming for at least 50% of our final participants to identify as female and for at least 20% to be visible minorities."* We are happy to confirm that we had 10 female participants (out of 21), and that we exceeded our target for visible minorities.

We also wrote: *"We will select a diverse and representative subset of the more senior participants to give longer talks"*. We are happy to say that we believe we achieved this goal.

In addition, we achieved diversity in participants for career stage, type of instition, and global location, amongst other criteria.

6 Ongoing Collaborations and Conclusion

The majority of the week was spent working in small groups on the open problems mentioned above. Several groups made progress towards their problems and, although none were entirely solved during the week, many groups are continuing to collaborate to build off the initial progress made in Banff. It is worth noting that many participants were, as intended, working in new areas outside of their expertise, so it is not unsurprising that progress will take more than a week to come.

We note, in addition, that a number of other collaborations (that is, to work on problems not discussed at the workshop) and research visits been planned by participants who met at our workshop.

We believe that the workshop was a great success, both scientifically and for the career enhancement of more junior researchers. We are very happy to have been able to facilitate such a positive impact on the more junior members of our community, especially given how disruptive the previous few years have been. We very much hope to be able to repeat this success with a similar event in the future.