WIN6: Women in Number Theory 6 (23w5175)

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1 Overview of the Workshop

The Women in Numbers (WIN) network was created in 2008 for the purpose of increasing the number of active women researchers in number theory. For this purpose, WIN sponsors regular conferences, taking place approximately every three years, where women scholars gather to collaborate on cutting-edge research in the field and produce publishable scientific results. The WIN workshops provide an ongoing forum for involving each new generation of junior faculty and graduate students in state-of-the-art research in number theory.

WIN6 workshop was organized in partnership with the Clay Mathematics Institute and held in Banff International Research Station, in Alberta, Canada, from March 26 to March 31, 2023. WIN6 was unique in the sense that the previous scheduled meeting was canceled due to the global pandemic. It was essential to pay careful attention in selecting participants and try to include those who were affected more significantly by travel limitations. In order to help raise the profile of active women researchers in number theory and increase their participation in research activities in the field, this event brought together researchers in different career stages and from broad subfields of number theory for collaboration. Emphasis was placed on on-site collaboration on open research problems as well as training junior number theorists. Both the project leaders and the participants were selected through a competitive application process. These were advertised through the WIN mailing list, as well as various other mailing lists for number theorists. There were 14 applications for project leadership (submitted by two senior mathematicians each) from which 11 groups were selected. Then we solicited applications for the participant pool, for which 71 applications were received. These applicants were asked to specify their preferences, and to explain their mathematical backgrounds. As most of these applicants would have been a great fit for the conference, we prioritized the career stages of the applicants in addition to their mathematical fit – the organizers agreed that postdocs and graduate students towards their final year of PhD are most likely to benefit from this conference, and as such, most of the participants who were invited were within this demographic.

The main goals were

1. To train graduate students and postdocs in number theory and related fields;
2. To highlight research activities of women in number theory;
3. To increase the participation of women in research activities in number theory;
4. To build a research network of potential collaborators in number theory;

5. To enable women faculty at small colleges to help advising graduate students.

We had a total of 42 participants, three of whom participated remotely due to not receiving a Canadian visa in time.

At BIRS, each group gave a brief 20-minute presentation on the topic of their research to facilitate conversations between different groups, and the majority of the remaining time was devoted to working in groups. In the evenings, we had various informal gatherings. Some were more structured in the form of a career panel, and others were more social and spontaneous.

Besides BIRS for providing excellent research facilities, we are grateful to the the following organizations for their support of this workshop: NSF, Number Theory Foundation, PIMS, Journal of Number Theory, and CMI.

2 Research Projects and Project Groups

In this section, we summarize the research topic and the progress of each project group, as reported by the project leaders of ten participating groups.

2.1 Campana points of bounded heights on orbifold stacks

Group members

Shabnam Akhtari, Jennifer Park, Marta Pieropan, Soumya Sankar

Project description

Recent work of Ellenberg, Satriano and Zureick-Brown [ESZ] has stirred a lot of interest towards the problem of counting rational points of bounded height on stacks. Special attention has been given to weighted projective stacks [Dar, BrMa], in particular, seen as moduli spaces of elliptic curves [BrNa, Phi], to modular curves [BoSa] and to stacky curves [NaXi]. The paper [ESZ] defines heights on stacks outlines an explicit connection between Malle's conjecture for number fields of bounded discriminant and Manin's conjecture for rational points of bounded height. Manin's conjecture has been extended to stacks by Darda and Yasuda [DaYa22] who also study Malle's conjecture as counting points of bounded height on classifying stacks [DaYa, DaYa2]. At the same time, there has been a lot of work on asymptotics of Campana points of bounded height. Campana points are rational points on a projective variety with prescribed intersection with a boundary divisor specified by some parameters. As the parameters vary, the sets of Campana points interpolate between the set of rational points and the set of integral points with respect to the same boundary divisor. Manin's conjecture for rational points has been extended to Campana points by Pieropan, Smeets, Tanimoto and Varilly-Álvarado [PSTV], and it has been proven for equivariant compactifications of vector groups [PSTV] and of Heisenberg groups [Xia], for toric varieties [PiSc, San], for various diagonal hypersurfaces in projective spaces [Van, BrVa, BrYa, Shu21, Shu20, BBKOPW] and for some non-strict-normal-crossing boundaries [Str]. This Women in Number Theory 6 project aims to explore the concept of Campana points on stacks. The leading goals are: identify a good notion of Campana points on stacks, compute asymptotic formulas in some examples, extend the Manin-type conjecture for Campana points [PSTV] to stacks, and investigate the relation to Malle's conjecture.

Progress

During the conference we considered two possible definitions of Campana points on stacks and we brainstormed several research directions. As result we collected at least six concrete projects that we can pursue in the future. The most immediate one consists of clarifying the relationship between Malle's conjecture for a finite group G and counting rational points of bounded height on BG. We made explicit calculations in the case where G is a cyclic group and established that the two problems coincide only in the case when the base field contains all $|G|$-roots of unity.

Future plans

We will carry on some of the projects we collected concerning counting Campana points of bounded height on specific stacks. We plan to work in parallel on a shared .tex file and to meet roughly once per month to monitor progress and share further ideas.

2.2 Generalized Eckardt points on del Pezzo surfaces of degree 1

Group members

Julie Desjardins, Yu Fu, Kelly Isham, Rosa Winter

Project description

Del Pezzo surfaces can be classified by their degree, which is an integer between 1 and 9. Those of degree 3 (which are smooth cubic surfaces in $\mathbb{P}^3$) famously contain 27 lines (exceptional curves) over an algebraically closed field, and at most 3 lines go through one point, called an Eckardt point. A cubic surface contains at most 45 Eckardt points. Similarly, a del Pezzo surface of degree 2 contains 56 lines over an algebraically closed field, at most 4 lines go through one point, and such a surface contains at most 126 such points. In this project we study the situation for del Pezzo surfaces of degree 1. These contain 240 exceptional curves, and we know that outside characteristics 2 and 3, at most 10 go through one point. There exist a handful of examples of such surfaces with a point contained in the intersection of 10 exceptional curves, but apart from that, very little is known.

Questions we aim to answer are: if 10 exceptional curves intersect in a point, which configurations can these curves have? How many 'Generalized Eckardt Points' (GEPs) can a del Pezzo surface of degree 1 contain? Can we produce examples of surfaces with multiple GEPs?

An important motivation for studying generalized Eckardt points on a del Pezzo surface of degree 1 comes from the search for proof of unirationality of these surfaces – which is a big open problem with only partial results. Del Pezzo surfaces of degree at least 3 are known to be unirational if they contain a rational point, with the extra condition that it is not a generalized Eckardt point for degree 2. Several methods exist for showing the density of the set of rational points on a del Pezzo surface of degree 1 (which is implied by unirationality), given a point which is not contained in many exceptional curves. It is not yet known how to prove similar results starting with generalized Eckardt points.

Progress

During the conference in BIRS we made a great start in classifying the possible configurations in which 10 ecxeptional curves on a del Pezzo surface of degree 1 can intersect. We have a list of 9 potential configurations, two of which are known to be possible, and 1 has been disproved. We are going over the remaining 6 configurations to either find examples that show they are possible or prove that they are not. We have also constructed a whole family of surfaces with 10 exceptional curves intersecting in a point, showing that certain surfaces always have a GEP.

Future plans

In the coming months we will work out the computations that we started with at BIRS. We plan to meet at the end of May to see where we are with the project and make a plan on how to continue working further, with regular meetings online. We want to write a paper for the proceedings, as well as keep working on the project afterwards, since there are still many open questions.

2.3 Patching Technique and Local Global Principles

Group members

Parimala Raman, Sujatha Ramdorai, Sarah Dijols, Charlotte Ure

Project description

The classical local global principle (also called Hasse Principle) for quadratic forms asserts that a quadratic form q over a global field has a nontrivial zero if and only if it has a nontrivial zero over all the completions. Such local global principles have subsequently expanded to other contexts:–to name a few, in studying the existence of rational points over varieties defined over global fields, non-triviality of Brauer group elements, nontriviality of torsors of algebraic groups, etc. Around two decades ago, Harbater and Hartmann introduced the patching technique to study algebraic structures over a semiglobal field F, which by definition is the function field $K(C)$ of a smooth integral curve C defined over a complete discretely valued field K. This method has proved fruitful in studying a variety of local global principles over a semi-gloobal field. Recent results of Harbater-Hartmann-Krashen use these methods to study the Galois cohomology groups of semiglobal fields.

Let L be a field and let l be a prime different from the characteristic of L. Let μ_l be the group of l-th roots of unity viewed as a module over the absolute Galois group of L for a field L. We assume that L contains μ_l. Let $\bar{L}$ denote a separable closure of L. Denote the absolute Galois group $\mathrm{G}al(\bar{L}/L)$ by G_L. The Galois cohomology groups $H^i(L, \mu_l) := H^i(G_L, \mu_l)$ for $i \geq 0$ encode important information on the arithmetic of L. A celebrated conjecture of Milnor, settled by Voevodsky, asserts that the groups $H^i(L, \mu_l)$ for $i \geq 2$ are generated by cup- products $(a_1) \cdot (a_2) \cdot \cdots \cdot (a_i)$, where (a_i) denotes the class of an element $a_i \in L^*$ under the Kummer isomorphism $H^1(L, \mu_l) \simeq L^*/L^{*l}$. Elements of the form $(a_1) \cdot (a_2) \cdot \cdots \cdot (a_i)$ are called *symbols*. The symbol length for $H^k(L, \mu_l)$ for $k \geq 2$ is the least number d such that any $\xi \in H^k(L, \mu_l)$ can be expressed as a sum of at most d symbols. For a field L, let $cd(L)$ denote the cohomolgocal dimension of L. The symbol length is intimately connected to other important invariants associated to a field such as the cohomological dimension and the u-invariant.

Conjecture Let F be a semiglobal field. With the notation as above, suppose that the cohomological dimension of k is n. Then every element of $H^{n+2}(F, \mu_l)$ is a symbol.

Goals of the project:

- Investigate symbol lengths for Galois cohomology groups of general semi-global fields
- Investigate applications of Patching Technique to study the conjecture above .

Progress

Some important reductions to the general setting were made. This would enable progress towards understanding symbol lengths.

Future plans

The results are being written up and we plan to continue the project discussion over zoom meetings.

2.4 Isogenous Discriminant Twins over Number Fields

Group members

Alyson Deines, Asimina Hamakiotes, Andreea Iorga, Changningphaabi Namoijam, Manami Roy and Lori Watson

Project description

The conductor of an elliptic curve E defined over a number field K is an arithmetic invariant. In particular, the conductor is an integral ideal in K that measures the ramification in the field extensions generated by the torsion points of E. The minimal discriminant is a geometric invariant; it measures the number of irreducible components of E over finite fields. The two invariants are closely related; the primes that divide the conductor are precisely the primes that divide the minimal discriminant. When two elliptic curves have the same conductor and minimal discriminant, they are called discriminant twins. One can ask, how many discriminant twins are there over a number field? In fact, one can start with an easier question, when can two prime-isogenous elliptic curves defined over a number field have the same minimal discriminant? Isogenous elliptic curves already have the same conductor, so this is asking, when can isogenous elliptic curves have the same discriminant?

Over the rationals there are only finitely many semistable isogenous discriminant twins [Dei18]. In particular, isogeny classes of size two with at least one prime with semistable reduction cannot have discriminant twins. In a recent work [BBDHR], the p-isogenous discriminant twins for $p = 3, 5, 7, 13$ over number fields have been classified. The number of j-invariants for the p-isogenous discriminant twins with $p = 3, 5, 7, 13$ over number fields is not necessarily finite. The results relies on studying an explicit parametrized family of elliptic curves for isogeny graphs of rational elliptic curves from [Bar].

Our planned project is to finish categorizing all isogenous discriminant twins over number fields. We can look at a few different questions:

1. Use parametrized family from [Bar] and the technique from [BBDHR] to get explicit result about composite isogeny degree of genus 0.

   i) For degree of isogeny $n = 9, 25$, a similar argument should follow.

   ii) The case when 2 divides the degree of isogeny is more complicated and requires more careful investigation.

2. Classify semistable discriminant twists over number fields.

3. Consider $n = 11, 17, 19, 21$, i.e., when $X_0(n)$ has genus one. Parameterizing the discriminant twins in this case is an open problem that will depend at least on the rank of the elliptic curve, $X_0(N)$, over a given number field.

Progress

During the workshop at Banff we started working on question (1) stated above for $n = 25$ and $n = 2$. We have spend some significant time discussing how to write some of these programs in Sage and checking examples in Sage for arriving at correct conjectures. We have the result for $n = 25$ and some partial results for $n = 2$. We have also discussed about how to approach some other cases like $n = 9, 6$.

Future plans

We have divided into subgroups working on different questions. We are meeting biweekly over Zoom as the whole group. In between subgroups are are meeting as needed. We plan to continue meeting over Zoom. We didn't plan any in-person meetings over Summer but some of us hope to attend the same conference in Summer.

2.5 Machine learning and arithmetic

Group members

Kristin Lauter, Cathy Li (participated remotely), Krystal Maughan, Rachel Newton, Megha Srivastava

Project description

We will explore and compare different machine learning approaches to arithmetic problems, some with relevance to cryptography.

Progress

During the WIN6 workshop, we experimented with different machine learning approaches and started building, training and testing a transformer model.

Future plans

We have arranged a regular weekly meeting slot.

2.6 Moments of Artin–Schreier covers

Group members

Alexandra Florea, Edna Jones, Mathilde Lalín

Project description

Let q be a power of an odd prime p. An Artin–Schreier cover is given (up to $\mathbb{F}_q$-isomorphism) by an affine model

$$C_f : y^p - y = f(x),$$

where $f(x) \in \mathbb{F}_q(x)$ is a rational function, together with the automorphism $y \mapsto y+1$. Artin–Scherier covers are interesting since they provide extreme cases of the Weil bound [RLW11].

By Weil's conjectures, the zeta function of C_f can be written as

$$Z_{C_f}(u) = \frac{P_{C_f}(u)}{(1-u)(1-qu)},$$

where $P_{C_f}(u)$ is a polynomial of degree $2g = (p-1)(\Delta - 1)$ where Δ codifies the order of the poles of $f(x)$. It can be written using additive characters of $\mathbb{F}_p$ as follows

$$P_{C_f}(u) = \prod_{\psi \neq \psi_0} L(u, f, \psi),$$

where the ψ_k, $k = 0, \ldots, p-1$ are the additive characters of $\mathbb{F}_p$ given by

$$\psi_k(a) = e^{2\pi i k a/p}, \quad k = 0, \ldots, p-1.$$

The moduli space of Artin–Schreier covers has a stratification given by the cardinality of the p-torsion of the Jacobian. In this context, there are some particular families that have been considered in the literature: $\mathcal{AS}_{g,0}$ corresponding to the case where $f(x)$ is a polynomial; $\mathcal{AS}_{g,p-1}$ corresponding to the case where $f(x)$ is a Laurent polynomial; $\mathcal{AS}_{g,g}$ (appearing when $(p-1) \mid g$) corresponding to the ordinary locus. Another family that has been considered is $\mathcal{AS}_{g,0}^{\text{odd}}$ corresponding to the odd polynomials.

The goal of this project is to study the moments of L-functions attached to Artin–Schreier covers, which can be considered without absolute value

$$\frac{1}{|\mathcal{AS}_{g,*}|} \sum_{f \in AS_{g,*}} L(1/2, f, \psi)^k$$

and with absolute value

$$\frac{1}{|\mathcal{AS}_{g,*}|} \sum_{f \in AS_{g,*}} |L(1/2, f, \psi)|^k,$$

where $\mathcal{AS}_{g,*}$ indicates any of the families described above. A motivation for this is the following: previous studies of mesoscopic statistics of the zeros showed these families to be indistinguishable [BDFL10], while more recent studies on n-level density showed these families to have different behaviour [EP]. One expects the zeros to be modeled by certain ensembles of random matrices depending on the specific subfamily under consideration. One should be able to distinguish the random group by studying moments.

Progress

We employed a description of the L-functions in terms of multiplicative characters due to Entin [Ent12] and we were able to compute the moments

$$\frac{1}{|\mathcal{AS}_{g,0}|}\sum_{f\in\mathcal{AS}_{g,0}} L(1/2,f,\psi)^k.$$

for roughly $k < p+1$.

Future plans

We are working on the moment

$$\frac{1}{|\mathcal{AS}_{g,0}|}\sum_{f\in\mathcal{AS}_{g,0}} |L(1/2,f,\psi)|^2$$

as well as moments for $\mathcal{AS}_{g,0}^{\text{odd}}$. We are meeting weekly on zoom to continue with the project.

2.7 Non-archimedean arithmetic dynamics

Group members

Jacqueline Anderson, Emerald Stacy, Bella Tobin

Project description

The Mandelbrot set is the set of complex numbers c for which the critical orbit (i.e., the orbit of 0) for $f_c(z) = z^2 + c$ is bounded. We'll call such a map *post-critically bounded*, or PCB. A *Misiurewicz point* is a point in the Mandelbrot set for which the orbit of 0 is strictly preperiodic. These points form a dense subset of the boundary of the Mandelbrot set. In 1989, Tan Lei proved that the Mandelbrot set zoomed in on a Misiurewicz point c and the corresponding Julia set for f_c zoomed in on c at the same scale were asymptotically similar about c. Moreover, both are asymptotically self-similar as well.

While the Mandelbrot set and its fractal boundary in $\mathbb{C}$ inspire many questions and reveal detailed dynamical information about quadratic polynomials, the analogous object over the non-Archimedean field $\mathbb{C}_p$ is simply the p-adic unit disk. However, if one generalizes the notion of PCB polynomials to higher degrees d and looks at the case where $p < d$, little is known about the corresponding Mandelbrot set, and what is known about a previously-studied one-parameter family of cubic polynomials over $\mathbb{Q}_2$ suggests it is more akin to the complex setting, with intricate boundaries and self-similarity.

In this project, we aim to develop a stronger understanding of p-adic Mandelbrot sets for degree d polynomials when $p < d$.

Progress

We selected a new cubic family of polynomials to study and worked out many of the basic dynamical properties of this family. We wrote Sage code to explore which maps in this family belong to the Mandelbrot set and which do not, and we proved these results in some cases. We gathered enough evidence to suggest that there are similar properties for this family to the ones we know from the previously-studied family (self-similarity near a Misiurewicz point, for example), but further work needs to be done to understand this family more fully.

We also developed some goals for more general theorems to try to prove regarding Misiurewicz polynomials of any degree d for any prime $p < d$, and we brainstormed some strategies for tackling the proofs.

Future plans

We are having weekly Zoom meetings to continue our work and to prepare our proceedings article. We are currently working to complete our analysis of the cubic family that we studied to better understand the patterns that appear in the Mandelbrot set, and we are also exploring tools to use to prove more general statements about critical orbit behavior of polynomials near Misiurewicz points.

2.8 Parametrizing moduli spaces of Artin-Schreier curves

Group members

Juanita Duque Rosero, Heidi Goodson, Elisa Lorenzo García, Beth Malmskog, Renate Scheidler

Project description

Elliptic curve isomorphism classes are given by the their j-invariants. In characteristic different than 2 and 3, an elliptic curve can be written as $y^2 = x^3 + ax + b$ and the j-invariant is given by $j = 1728\frac{4a^3}{16(4a^3+27b^2)}$. In characteristic 2 and 3, the formula looks much more complicated. The isomorphism classes of curves of genus 2 are given by the Igusa invariants J_2, J_4, J_6, J_8 and J_{10}. The invariant J_8 is only needed in characteristic 2.

Curves of genus 3 can be hyperelliptic or non-hyperelliptic. For hyperelliptic curves there are in general algorithms for computing their invariants in characteristic 0 by using the ”transvectant construction”. This construction also works in characteristic p for p big enough. In particular, for genus 3 it works for characteristic greater than 7. The remaining cases were treated by Basson in his thesis and some later unpublished work. In characteristic different from 2, the strategy is very similar to the one in characteristic 0, we just need to be careful with the denominators appearing in the computations. The characteristic 2 case needs a completely different approach: in this case these curves are Artin-Schreier curves.

The isomorphism classes of non-hyperelliptic curves of genus 3 are given by the Dixmier-Ohno invariants in characteristic 0. The extra needed invariants for all characeristics except for characteristic 3 are computed in work by Liu-Lercier-Lorenzo García-Ritzenthaler. The problem curves here for the standard method are the Picard curves, given by an equation $y^3 = f(x)$ with $\deg(f) = 4$ in characteristic different from 3 and Artin-Schreier curves in characteristic 3.

For every characteristic except $p = 3$, a stratification by automorphism groups of non-hyperelliptic curves of genus 3 curves is known, see the work of Ritzenthaler and others. Again the obstruction in characteristic 3 is the class of Artin-Schreier curves of genus 3.

In order to deal with these remaining cases we propose to study invariants of Artin-Schreier curves. The first ingredient needed for this is understanding the isomorphisms between Artin-Schreier curves, see Franell thesis.

The proposed project is the following:

1. Describe isomorphism between Artin-Schreier curves.
2. Give an algorithm to compute invariants for a given genus g and characteristic p.
3. Compute the invariants for small values. In particular for $(g, p) = (3, 3)$.
4. Give an algorithm to compute automorphisms of Artin-Schreier curves.
5. Compute the automorphisms for small values. In particular for $(g, p) = (3, 3)$.

Progress

We described the isomorphisms between Artin-Scheier curves, we worked out meaningful examples for the invariants computation ($p = 3$ and $g = 3$, and any p and 4 poles of equal multiplicity to 1) and we sketched a general algorithm to do so.

Future plans

We are meeting every 2 weeks on Zoom. We are very optimistic about getting a paper with all our results ready for the proceedings deadline on 15th January.

2.9 Residue fields of algebraic points on curves in a linear system

Group members

Irmak Balçik (participated remotely), Stephanie Chan, Yuan Liu (participated remotely), Bianca Viray

Project description

Let $f\colon Y \to X$ be a finite map of smooth projective geometrically integral curves (not necessarily étale) defined over a number field K, and let d be the degree of the morphism f. Given any $x \in X(K)$ outside of the ramification locus of f we obtain a degree d étale algebra over K by taking the product of residue fields

$$\mathbf{k}(Y_x) := \prod_{y\in Y, f(y)=x} \mathbf{k}(y).^1$$

Our broad goal is to obtain information about the family of all degree d étale algebras that arise in this way in the case when $X(K)$ is infinite. In the particular case when $X = \mathbb{K}^1$, then Hilbert's irreducibility theorem states that $\mathbf{k}(Y_x)$ will be a degree d field extension for infinitely many points $x \in \mathbb{P}^1(K)$, so a particularly interesting case is to study constraints on the degree d field extensions that arise in this way.

If Y has positive genus and the Jacobian of Y has rank 0, then all but finitely many degree d extensions L/K over which $Y(L) \supsetneq \cup_{L/F/K, L\neq F} Y(F)$ arise as the fiber of such a map. Thus understanding this question gives us complete information about all but finitely many degree d points on Y.

Progress

Over the course of the WIN6 workshop, we completely answered this question for superelliptic curves over a nonarchimedean local field. Namely, if $f\colon Y \to \mathbb{P}^1$ is degree d morphism of curves over a nonarchimedean local field K where Y is given by an equation of the form $y^d = g(x)$ for a separable polynomial g and f, Y and X have good reduction, then we have determined which degree d étale algebras L/K arise from fibers of f. Moreover, we computed the Haar density of the locus in $\mathbb{P}^1(K)$ that gives rise to a specific étale algebra. In addition, we have made progress in extending these results from the case of superelliptic curves to an arbitrary degree d cover of $\mathbb{P}^1$.

Future plans:

We are currently in the process of writing up our results on superelliptic curves. In May, we plan to resume regular meetings to work out the case of arbitrary covers and determine other results that we may wish to pursue.

2.10 Large Sums of Fourier Coefficients of Cusp Forms

Group Members

Claire Frechette, Mathilde Gerbelli-Gauthier, Alia Hamieh, and Naomi Tanabe.

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[1] We can also view $\mathbf{k}(Y_x)$ as the ring of global sections of the sheaf of total quotient rings of Y_x. This is the generalization of "residue field" for a variety that is not itself geometrically integral, but that is a union of geometrically integral varieties.

Project description

Let χ be a primitive character $\mod q$. An important problem in analytic number theory is establishing the asymptotic $\sum_{n\leq x}\chi(n)=o(x)$ for as wide a range of x as possible. In [GS1], Granville and Soundararajan proved that under the GRH, the character sum $\sum_{n\leq x}\chi(n)=o(x)$, as $\frac{\log x}{\log\log q}\to\infty$. They also proved that this range of x is the best possible. In [Lam], Lamzouri considered the analogous problem for large sums of the normalized Fourier coefficients of a holomorphic Hecke cuspform of weight k, as $k\to\infty$. See [Lam, Corollary 1.2, Corollary 1.4] for the statements of his results. In [GS2], Granville and Soundararajan established more concrete connections between large character sums and zeros of $L(s,\chi)$ which allowed them to establish the following result

Theorem 2.1. *Let $\chi\ (\mathrm{mod} q)$ be a primitive character. Let ϵ and T be real numbers with $1\leq T\leq(\log q)^{\frac{1}{200}}$ and $\epsilon\geq(\log q)^{-\frac{1}{3}}$. Suppose that for every real τ with $|\tau|\leq T$ the region*

$$\left\{s\,:\,\Re(s)\geq\frac{3}{4},|\Im(s)-\tau|\leq\frac{1}{4}\right\}$$

contains no more than $\epsilon^2(\log q)/1440$ zeroes of $L(s,\chi)$. Then for all $x\geq q^\epsilon$ we have

$$\left|\sum_{n\leq x}\chi(n)\right|\ll\frac{x}{T}.$$

In this project, we aim to establish analogues of Theorem 2.1 for sums of Fourier coefficients of an automorphic cusp form π of large analytic conductor $C(\pi)$. We would like to prove that $\sum_{n\leq x}\lambda_\pi(n)=o(x\log x)$ in a wide range of x (perhaps $x>C(\pi)^\epsilon$) under a weaker assumption than the GRH. In this work, we will assume that the family of automorphic forms π satisfies the Generalized Ramanujan Conjecture to ensure that the Fourier coefficients under consideration are bounded by a divisor function.

Progress

The work in [GS2] relies heavily on deep results on mean values of multiplicative functions such as the improved versions of Halász's theorem and the Lipschitz-type estimates established in [GS3]. Such results have been extensively studied in the last 2 decades for 1-bounded multiplicative functions by Granville, Harper, Koukoulopoulos, Matomaki, Radziwill and Soundararajan. However, it was not until recently that generalizations of such results have been pursued for multiplicative functions that are not 1-bounded. Of particular interest to us is the work of Mangerel [Man23] and [GHS]. To establish our goal, we need mean value theorems for multiplicative functions that are bounded by the divisor function. Although [Man23] and [GHS] lay out much of the foundational work in this direction, few results that are crucial to our work are not available in literature in the specific form that we need. During WIN6 workshop, our group worked on collecting and establishing all the preliminary results and lemmas needed to prove our main theorem. In order to gauge the difficulty of the problem, we focused first on Fourier coefficients of a classical cusp form f for $\mathrm{SL}_2(\mathbb{Z})$ of large weight k. Indeed, our efforts were successful, and we managed to obtain all the results needed to establish the desired upper bound for $\sum_{n\leq x}\lambda_f(n)=o(x\log x)$ under certain conditions on the distribution of zeros of the associated L-function $L(s,f)$ in the critical strip.

Future plans

Our team plans to meet biweekly on zoom during the summer. We might plan a hybrid meeting during the summer. Our initial short term goal (i.e. to be accomplished within a month) is to put together all the results that we have established thus far to finalize the proof of our main theorem in the special case of classical cusp forms that are varying in the weight aspect only. In doing so, we follow the general framework employed in [GS2]. The next goal would be to establish these results in the more general setting of cuspidal automorphic forms over $\mathrm{GL_d}(\mathbb{A}_\mathbb{Q})$ that are varying in the analytic conductor aspect.

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