# Algebraic Aspects of Matroid Theory 23w5149

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March 12 – March 17, 2023

This workshop brought together international mathematicians working in the areas of matroid theory, algebra, algebraic geometry, and combinatorics at large. The workshop focused on three main areas of research in connection with matroid theory: log-concavity and combinatorial Hodge theory, matroids over hyperfields, and connections to tropical geometry. The scientific activities of the workshop included 10 one hour long talks, 12 talks of 20 minutes by junior participants, as well as open problem, discussion, and Q&A sessions. Here we present an overview of the field, open problems arising from the meeting, as well as a report on workshop outcomes and work in progress.

## 1 Overview of the Field

Matroids are beautiful and important objects which lie at the interface of combinatorics and geometry, and in recent years some sophisticated algebra has appeared in connection with matroid theory as well. Originally introduced independently by Hassler Whitney [21] and Takeo Nakasawa [17], matroids are a way of simultaneously axiomatizing the notion of linear independence in vector spaces and the notion of acyclicity in graphs. Here are three topics of current interest which blend algebra and matroid theory in new and exciting ways:

### 1.1 Log-concavity and combinatorial Hodge theory

Read conjectured in 1968 that for any finite graph $G$, the sequence of absolute values of the coefficients of the chromatic polynomial of $G$ is log-concave (and in particular unimodal). Read's conjecture was proved by June Huh in 2012 using methods from algebraic geometry [13]. Huh's result was subsequently refined and generalized from graphs to matroids in 2015 by Adiprasito, Huh, and Katz, settling a longstanding conjecture of Rota [1]. In the process, these authors found a purely combinatorial incarnation of Hodge theory, establishing analogs in matroid theory of the Hard Lefschetz Theorem and Hodge-Riemann relations in algebraic geometry.

In the last few years, the methods of Huh et al. have been used by several sets of authors to establish log-concavity of various sequences arising naturally in combinatorics and geometry, e.g., certain Littlewood-Richardson coefficients occurring in representation theory and the number of independent subsets of a vector space over a finite field. New methods, closely related to Hodge theory for matroids, have recently been developed for such problems using the theory of Lorentzian polynomials, as introduced by Brndn and Huh [9]. This same class of polynomials was independently studied by Anari, Liu, Oveis Gharan, and Vinzant [3].

Those authors also relate these polynomials to the mixing times of certain Markov chains / high-dimensional random walks and efficient algorithms for approximating the number of bases of a matroid.

## 1.2 Matroids over partial hyperstructures

One of the most important questions combinatorialists ask about a matroid $M$ is over what fields $M$ is representable. More generally, one can ask about representability over partial fields. Pendavingh and van Zwam introduced the universal partial field of a matroid $M$, which governs the representations of $M$ over all partial fields [18]. Unfortunately, asymptotically 100% of matroids are not representable over any partial field, and in this case, the universal partial field gives no information.

Using the recently introduced theory of matroids over partial hyperstructures developed by Baker and Bowler [6] (itself rooted in the work of Dress and Wenzel), Baker and Lorscheid have introduced a significant generalization of the universal partial field which they call the foundation of a matroid [7]. The foundation of $M$ is a new type of algebraic object which Baker and Lorscheid call a pasture; examples of pastures include both partial fields and hyperfields in the sense of Krasner. Pastures form a natural class of "field-like" objects within Lorscheid's "ordered blueprints" (which were originally introduced in connection with geometry over the "field of one element"), and they have desirable categorical properties (e.g., existence of products and coproducts) that make them an appealing new context in which to study algebraic invariants of matroids. The foundation of $M$ occurs naturally as the "residue pasture" of the point of the moduli space of matroids (an ordered blue scheme constructed by Baker and Lorscheid) corresponding to $M$.

The foundation of a matroid $M$ represents the functor taking a pasture $F$ to the set of rescaling equivalence classes of $F$-representations of $M$; in particular, $M$ is representable over a pasture $F$ if and only if there is a homomorphism from the foundation of $M$ to $F$. Pastures have nicer categorical properties than partial fields, which allows one to state and prove results such as "the foundation of a direct sum of matroids is the tensor product of the foundations". We expect this formalism to lead to an improved understanding of matroid representations over fields, especially the interaction between such representations and notions such as orientability, unique orientability, and unique representability.

## 1.3 Matroids and tropical geometry

Matroids play an essential role in tropical geometry: for example, they encode tropical linear spaces via the Bergman fan construction [4]. (Technically speaking, a tropical linear space is the Bergman complex associated to a valuated matroid, which incidentally is the same thing as a representation of a matroid over the tropical hyperfield). The tropical varieties which behave most like a smooth complex manifold $X$ are those which look locally like the Bergman fan of a matroid at every point (such tropical varieties are therefore called smooth). For smooth tropical varieties, Itenberg, Katzarkov, Mikhalkin, and Zharkov recently developed theory of tropical cohomology which encodes how Hodge cohomology groups $H^{p,q}(X, \mathbb{C})$ degenerate in one-parameter families of smooth complex varieties [14]. One construction of tropical cohomology is based on the Orlik-Solomon algebra associated to a matroid, which was originally introduced to compute the singular cohomology of the complement of a complex hyperplane arrangement [22]. The development of tropical cohomology opens the door to new applications of tropical geometry to both real and complex geometry.

Bergman fans of matroids are also intimately connected to tropical intersection theory, as shown in the pioneering work of Allerman, Rau, and Shaw [2], [20]. Brugallé and Shaw have provided intersection theoretic obstructions to lifting tropical curves on tropical surfaces to algebraic curves on algebraic surfaces based on positivity properties in the algebraic situation [10]. Although the Hodge index theorem holds on matroids by Adiprasito, Huh, and Katz, it fails on more general smooth tropical surfaces as shown by Shaw [19]. The Hodge index theorem can also fail in the case of non-matroidal fans and has been used by Babaee and Huh to give a counterexample to Demailly's conjectured strengthening of the Hodge Conjecture in algebraic geometry [5].

## 1.4 Additional topics

Other topics of current interest include:

1. Frobenius flocks and algebraic matroids

2. Kazhdan-Lusztig polynomials of matroids
3. Free resolutions of matroid ideals
4. Positively hyperbolic varieties and positroids

# 2 Presentation Highlights

The workshop featured the following hour-long talks. They are presented here in sequential order.

**Federico Ardila** spoke about "Combinatorial Intersection Theory: A Few Examples".

**Abstract:** Intersection theory studies how subvarieties of an algebraic variety X intersect. Algebraically, this information is encoded in the Chow ring A(X). When X is the toric variety of a simplicial fan, Brion gave a presentation of A(X) in terms of generators and relations, and Fulton and Sturmfels gave a "fan displacement rule to intersect classes in A(X), which holds more generally in tropical intersection theory. In these settings, intersection theoretic questions translate to algebraic combinatorial computations in one point of view, or to polyhedral combinatorial questions in the other. Both of these paths lead to interesting combinatorial problems, and in some cases, they are important ingredients in the proofs of long-standing conjectural inequalities. This talk will survey a few problems on matroids that arise in combinatorial intersection theory, and a few approaches to solving them. It will feature joint work with Graham Denham, Chris Eur, June Huh, Carly Klivans, and Ral Penaguio.

**Oliver Lorscheid** spoke about "Categories of matroids and matroid bundles".

**Abstract:** Baker and Bowler's theory of matroids with coefficients can be understood as an extension of linear algebra from fields to unwieldier objects such as partial fields and hyperfields. In this talk, we complement this theory with the notion of a morphism of matroids with coefficients, which passes through a subtle process that we call "perfection". Eventually we gain a categorical framework for matroid bundles over F1-schemes. All this stems from joint ideas with Baker, Jarra and Jin. As a sample application, we define the Tutte-Grothendieck ring of an F1-scheme, which can be seen as a "detropicalization" of algebraic K-theory. The Tutte-Grothendieck ring of the moduli space of matroids carries a "universal Tutte class" whose pullback to any matroid is the Tutte polynomial of the matroid. If time allows, we muse about how this might be used to reprove the Fink-Speyer theorem.

**Chris Eur** spoke about "How or when do matroids behave like positive vector bundles?".

**Abstract:** Motivated by certain toric vector bundles on a toric variety, we introduce "tautological classes of matroids" as a new geometric model for studying matroids. We describe how it unifies, recovers, and extends various results from previous geometric models of matroids. We then explain how it raises several new questions that probe the boundary between combinatorics and algebraic geometry, and discuss how these new questions relate to older questions in matroid theory.

**Shiyue Li** spoke about "$K$-rings of matroids".

**Abstract:** I will share some discoveries on K-rings of wonderful varieties and matroids. The main result is a HirzebruchRiemannRoch-type theorem. I will also discuss applications to moduli spaces of curves. Joint work with Matt Larson, Sam Payne and Nick Proudfoot.

**Nima Anari** spoke about "High-dimensional expansion and sampling algorithms: what lies beyond log-concave polynomials and matroids".

**Abstract:** I will survey high-dimensional expanders (HDX) and the alternative perspective they provide on some of the recent advances in matroid theory concerning log-concave/Lorentzian polynomials. The HDX perspective has been key in solving algorithmic problems concerning sampling and/or counting in combinatorial structures, including matroids and some objects beyond matroids (such as matchings, Eulerians tours, etc.). I will formulate conjectures which, if proven, would generalize parts of the theory that has been developed for log-concave polynomials/matroids. I will then mention some results concerning fractionally log-concave and sector-stable polynomials, which provide evidence for the general conjectures.

**Omid Amini** spoke about "Hodge theory for tropical fans".

**Abstract:** I will present a proof of the Khler properties of the Chow ring for a large class of tropical fans based on three basic operations on fans which preserve the balancing condition (orientability). In the case of matroids, this allows to circumvent some of the difficulties arising in the work by Adiprasito, Huh, and Katz. Time permitting, I will discuss generalizations both in the local and global settings, and some applications to geometric questions. Based on joint works with Matthieu Piquerez.

**Nick Proudfoot** spoke about "Equivariant/Categorical Matroid Invariants".
**Abstract:** The characteristic polynomial of a matroid is categorified by the Orlik-Solomon algebra, and questions about the characteristic polynomial can be enriched to questions about the Orlik-Solomon algebra, now regarded as a graded representation of the group of symmetries of the matroid. Other polynomial invariants with natural categorifications include the Chow and augmented Chow polynomials, the Kazhdan-Lusztig polynomial, and the Z-polynomial. I will survey various results and conjectures about these categorical invariants, ending with a discussion of what it means for a categorical invariant to be valuative.

**Lucía López de Medrano** spoke about "Chern classes of tropical manifolds".
**Abstract:** In this talk, we will explain the extension of the definitions of Chern-Schwartz-MacPherson (CSM) cycles of matroids to tropical manifolds. With this definition, we will see a correspondence theorems for the CSM classes of tropicalisations of subvarieties of toric varieties, an adjunction formula relating the CSM cycles of a tropical manifold and a codimension-one tropical submanifold and a Noethers Formula for compact tropical surfaces. Joint work with Felipe Rincn and Kris Shaw.

**Alex Fink** spoke about "Matrix orbit closures and their classes".
**Abstract:** If an ordered point configuration in projective space is represented by a matrix of coordinates, the resulting matrix is determined up to the action of the general linear group on one side and the torus of diagonal matrices on the other. We study orbits of matrices under the action of the product of these groups, as well as their images in quotients of the space of matrices like the Grassmannian. The main question is what properties of closures of these orbits are determined by the matroid of the point configuration; the main result is that their equivariant K-classes are so determined. I will also draw connections to positivity and the work of Berget, Eur, Spink and Tseng. The results of mine featured here are mostly joint with Andy Berget.

**Diane Maclagan** spoke about "Tropical schemes - problems and progress".
**Abstract:** In this talk I will briefly describe the program to develop tropical schemes, with an emphasis on recent progress. Tropical schemes can be described as towers of (valuated) matroids, and I will focus on questions that arise at the inferface between the geometry and matroid theory.

We also had 20-minute talks by Tong Jin, Zach Walsh, Hunter Spink, Jacob Matherne, Benjamin Schroeter, Matt Larson, Colin Crowley, Anastasia Nathanson, Nick Anderson, Chi Ho Yuen, Ahmed Umer, and Tara Fife. In addition, the workshop featured three Open Problem discussions and two Q&A discussions.

# 3 Open Problems

Here we compile some open problems that were brought up during the community discussions in order to make this list open and accessible to the research community.

## 3.1 Log-concavity and combinatorial Hodge theory

1. Can one give a purely combinatorial proof of the log-concavity of Kostka numbers in representation theory? What about other instances of Okounkov's conjectures on Littlewood-Richardson coefficients?

2. Matroids can be thought of as the "Type A" case of the more general notion of Coxeter matroids, due to Gelfand and Serganova. What are the analogs of combinatorial Hodge theory and the theory of Lorentzian polynomials for other "Lie types" (corresponding to other families of finite Coxeter groups)?

3. The well-known "negative correlation" property for spanning trees in graphs does not generalize in a naive way to matroids, as there are subtle counterexamples. However, Huh-Schroter-Wang and Brndn-Huh have proved that "negative correlation" (which is equivalent to "1-Rayleigh") can be replaced

by “2-Rayleigh” in the case of matroids, and they conjecture that the optimal Rayleigh constant for matroids is 8/7. Can one further develop the theory of Lorentzian polynomials and combinatorial Hodge theory and in the process hone in on this conjecture?

4. **(Chris Eur)** Consider

$$A^\bullet(X_E)[\delta]/\langle \delta^r + \delta^{r-1}c_1(\mathcal{S}_M) + \cdots + c_r(\mathcal{S}_M)\rangle.$$

   The generator of the ideal is called a **Chern polynomial**. If $M$ is realized by a linear space $L$, then this ring $\simeq A^\bullet(\mathbb{P}(\mathcal{S}_L))$.

   **Question:** Do Hard Lefschetz and Hodge–Riemann hold for this ring with $l = c\delta + a$, for $a$ ample on $X_E$?

5. (**Matt Larson**) **Conjecture.** $T_M(x+1, x+1)$ has log-concave coefficients for all matroids $M$. True for $|E(M)| \leq 9$. Fact:

$$T_M(x+1, x+1) = \sum_{u \in \{0,1\}^n} x^{d(P(M),u)},$$

   where $d$ is the lattice distance.

6. **(Chris Eur)**: Given a matroid quotient $M \twoheadrightarrow N$, we have a canonical **Higgs factorization**

$$M \twoheadrightarrow M_1 \twoheadrightarrow M_2 \twoheadrightarrow \cdots \twoheadrightarrow N.$$

   Each successive quotient in the factorization comes from a matroid $\hat{M}_i$, where $M_{i-1}$ is a single-element deletion of $\hat{M}_i$ and $M_i$ is a single-element contraction of $\hat{M}_i$. Consider the beta-invariants of the matroids $\hat{M}_i$.

   **Question**: Is the sequence $\beta(\hat{M}_1), \ldots, \beta(\hat{M}_{r-1})$ log-concave?

7. (**Matt Larson**) Suppose $L$ is a linear subspace of $K^n$, where $K$ is a field, giving rise to a loopless matroid $M$ of rank $r$. Let $P$ be a full-dimensional generalized permutohedron in $\mathbb{R}^n$, and define $R^{\cdot}(P, L)$ to be the image of

$$\oplus_{k \geq 0} H^0(X_{A_{n-1}}, \mathcal{O}(kP))$$

   in $\mathbb{P}L \cap T$.

   **Conjecture 1:**

$$\dim R^k(P, L) = \chi(W_L, \mathcal{O}(kP)) = \chi(M, \mathcal{O}(kP)).$$

   This would follow if $H^i(W_L, \mathcal{O}(kP)) = 0$ for $i > 0$ and we have surjectivity on $H^0$.

   **Conjecture 2:** $R^{\cdot}(P, L)$ is a Cohen-Macaulay ring.

   **Conjecture 3:**

$$\sum_{k \geq 0} \chi(M, \mathcal{O}(kP))t^k = \frac{Q(t)}{(1-t)^r},$$

   where the coefficients of $Q$ are nonnegative.

   **Remarks:**

   (a) Conjecture 1 + Conjecture 2 implies Conjecture 3 if $M$ is realizable.

   (b) These are true if $L = K^n$, $P$ is the standard simplex, or $P$ is the negative of the standard simplex.

   (c) If true, these conjectures would show that $[t^r]g_M(t) \geq 0$.

### 3.2 Matroids over partial hyperstructures

1. There are, in fact, two types of representations, called weak and strong, for matroids over pastures; they coincide for perfect pastures (e.g., partial fields and doubly-distributive hyperfields). Is there a purely algebraic characterization of a class of perfect pastures which contains both doubly-distributive hyperfields and partial fields and is closed under taking products?

2. Can the moduli space of matroids be extended to other "types", e.g., to a moduli space of orthogonal or symplectic matroids? (This might be related to the theory of cluster algebras.)

3. Can the moduli space of matroids and the theory of pastures be used to study the (in)famous conjectures of Macpherson et al. regarding the homotopy type of the moduli space of oriented matroids? There is an intriguing analogy between the latter and the Dressian (a kind of "tropical Grassmannnian"), whose "combinatorial skeleton" is somewhat better understood than its oriented counterpart.

4. (**Oliver Lorscheid**) Linear spaces satisfy not just the usual Plücker relations but also **multi-exchange relations**: given bases $B, B'$ and a set $A \subset B \setminus B'$ of size $l$, there exists a set $A' \subset B' \setminus B$ of size $l$ such that $B \setminus A \cap A'$ and $B' \setminus A' \cap A$ are bases. It is also true that the single exchange relations implies the multi-exchange relations for matroids, i.e. over the Krasner hyperfield $\mathbb{K}$. Is the same true for all idylls? Or at least perfect ones?

### 3.3 Matroids and tropical geometry

1. Valuated matroids arise in the definition of tropical ideals due to Maclagan and Rincn, following the pioneering work on tropical scheme theory due to Giansiracusa and Giansiracusa. Every tropical ideal in the sense of Maclagan-Rincn has an associated tropical variety (a finite polyhedral complex equipped with positive integral weights on its maximal cells), and a basic question is which weighted polyhedral complexes arise in this manner. Using work of Las Vergnas on the non-existence of tensor products of matroids, Draisma and Rincn recently found a matroid whose associated tropical linear space does not come from a tropical ideal. Can one modify the definition of tropical ideal in a way which eliminates such counterexamples while retaining the useful properties (e.g., existence of Hilbert polynomials) of tropical ideals? The alternative theory of tropical schemes proposed by Oliver Lorscheid (based on his theory of ordered blueprints), along with the notion of families of matroids arising in the work of Baker and Lorscheid, could be useful in this context.

2. What is the relationship between tropical cohomology and the Adiprasito-Huh-Katz Chow ring of a matroid?

3. The Riemann-Roch theorem for tropical curves (due to Baker-Norine, Gathmann-Kerber, and Mikhalkin-Zharkov) currently has no cohomological formulation or proof. It seems natural to try to use the theory of ordered blueprints and families of matroids over such objects to develop a suitable cohomology theory in this context.

4. Lucía López de Medrano, Felipe Rincn, and Kris Shaw recently defined tropical Chern-Schwartz-MacPherson (CSM) cycles for an arbitrary matroid $M$. These are certain balanced weighted fans supported on the corresponding Bergman fan which have nice combinatorial properties, e.g. they "categorify" the reduced characteristic polynomial of $M$. Can tropical CSM cycles be used to formulate (and perhaps even prove) a higher-dimensional Riemann-Roch theorem in tropical geometry?

5. (**Hunter Spink**) June Huh and Eric Katz proved that $\deg(\Delta_M \cap \beta^{r-1} = T(1, 0)$, which counts the number of bases of external activity 0. This says that if we have a linear space $L \subseteq \mathbb{C}^n$ of dimension $r$ and we have an $(r-1)$-dimensional "reciprocal linear space" $\Lambda$, then $|L \cap \Lambda| = T_M(1, 0)$.

   On the tropical side, there is in fact a natural bijection (Katz, Berget–Spink–Tsend) between the tropical intersection points of $\Delta_M$ and $\beta^{r-1} + v$ and bases of external activity zero with respect to the chamber that $v$ lies in. (Note: the multiplicities are always 1.)

   **Question:** What happens when we cross a wall?

More precisely:

**Problem:** Prove that monodromy is transitive.

### 3.4 Matroid polytopes

(**Chris Eur**) Fact: if $P$ is a lattice generalized permutahedron, then $P \cap ([0,1]^n + v)$ is also, for $v \in \mathbb{Z}^n$. Tile $\mathbb{R}^n$ by cubes; this gives a decomposition of $P$ into translates of matroid polytopes.

**Problem:** Do this as explicitly as possible for graphical zonotopes,

$$Z_G = \sum_{(v_1,v_2)\in G} \mathrm{Conv}(e_{v_1}, e_{v_2}) \subset \mathbb{R}^{V(G)}.$$

### 3.5 Matroids of other Coxeter types

1. (**Chris Eur**) Let $\pi : \mathbb{R}^{2n} \overset{[I_n\ -I_n]}{\longrightarrow} \mathbb{R}^n$. A delta-matroid $D$ is **envelopable** if there exists a matroid $M$ on $[2n]$ such that $\pi(P(M)) = P(D)$, possibly with scaling depending on conventions. Not all delta-matroids are envelopable.

   **Question:** Are all even delta-matroids envelopable? Are all delta-matroids with the strong symmetric exchange property envelopable?

2. (**Matt Larson**) If $M$ is a matroid, the rank polynomial $R_M(u,0)$ gives the $f$-vector of the independence complex of $M$. Similarly, $R_M(u,-1)$ gives the $f$-vector of the non-broken circuit complex (or Orlik–Solomon algebra).

   Now let $D$ be a Delta-matroid. In this case, there is again a polynomial $U_D(u,0)$ which gives the $f$-vector of the independence complex of $M$.

   **Question:** What is the meaning of $U_D(u,-1)$?

   Let $U_D(u,-1) = f_0u^n + \cdots + f_n$.

   **Problem:** Show that $f_i \leq f_{n-i}$ for all $i \leq n/2$ for every Delta-matroid $D$.

   This would be true if the $f_i$'s were the $f$-vector of a pure simplicial complex.

3. (**Graham Denham**)

   **Problem #1**: Develop an activity theory for Delta-matroids.

   **Problem #2**:Is there an analogue of the Orlik–Solomon algebra for Delta-matroids?

### 3.6 Matroids and representation theory

1. (**Andy Berget**) **Conjecture.** The number of set partitions of $E(M)$ into independent sets of $M$ of sizes $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$, $\lambda \vdash |E(M)|$, is at least the Kostka number $K_{\lambda,\rho^t}$, where $\rho = \rho(M) : r_1 \geq r_2 \geq \cdots$ is the **rank partition** of $M$, determined by the condition that $r_1 + \cdots + r_k =$ size of the largest union of $k$ independent sets of $M$, i.e. the rank of the $k$-fold matroid union of $M$. (Assume $M$ is loopless.)

   Motivation. Pick a realisation $v_1, v_2, \ldots, v_+n \in \mathbb{C}^r$ of $M$. Form

   $$\mathfrak{S}(v) = \mathrm{span}(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} | \sigma \in \mathfrak{S}_n) \subset (\mathbb{C}^r)^{\otimes n}.$$

   This is an $\mathfrak{S}_n$-representation, so it decomposes into irreducibles, indexed by partitions. It's a consequence of [Berget–Fink] that the multiplicity of each irrep is a valuative matroid invariant.

   Theorem. The irrep indexed by $\lambda$ appears iff $\lambda \trianglerighteq \rho^t$, where $\trianglerighteq$ is dominance order.

Theorem. The multiplicity of $\lambda =$ a hook gives the coefficients of $\overline{\chi}_M$ up to sign.

The **Frobenius character** of a $\mathfrak{S}_n$-representation is its character written as a symmetric function.

**Variant conjecture.** The Frobenius character of $\mathfrak{S}(V) - e_{\rho^t}$ is Schur-positive. Here $e_{\rho^t}$ is an elementary symmetric function. The Gröbner degeneration $X(v) \rightsquigarrow$ in $X(v)$ from [Berget–Fink] should have a matroidal extension, and the Frobenius character should be computable from it.

2. (**Shiyue Li**) Let $V_\bullet^n$ be the $\mathbb{C}$-vector space spanned by the permutations in $S_n$. There is a natural $S_n$ action on this space by conjugation. This representation has a natural grading, where the $i^{\text{th}}$ graded piece consists of permutations with $i$ cycles.

   **Question 1:** (Gedeon–Proudfoot–Young): Is $V_\bullet^n$ equivariantly log-concave?

   Note that $\dim V_i^n$ is equal to the unsigned Stirling number $c(n, i)$ of the first kind, and this sequence is known to be log-concave.

   **Question 2:** Is the Poincaré polynomial of $V_\bullet^n$ "equivariantly real-rooted"?

### 3.7 Combinatorics of matroids

1. (**Johannes Rau**) This question is based on work in progress by Draisma, Pendavingh, Rau, Yuen, and a student of Draisma.

   Given a matroid $M$, we have inequalities between three numbers:

$$\begin{aligned} &d := \operatorname{rk}(M) \\ &\leq \min\{2\dim(\Sigma_M + R) - \dim R : R \text{ a rational subspace of } \mathbb{R}^n\} \\ &\leq \min\{\sum(2\operatorname{rk}_M(P_i) - 1) : P_1 \amalg \cdots \amalg P_k = E\}. \end{aligned}$$

   The third number is bounded above by $\min\{n, 2d-1\}$. The third number is the second specialized to $R$ being a subspace in the braid arrangement. For $M$ realizable over $\mathbb{C}$ by a subspace $V$, the second and third agree and both equal $\dim(\operatorname{Log}(V))$.

   **Question:** Are the second and third always equal?

   **Question:** Compute these three numbers for the restriction of $M$ to each set $S \subset E(M)$, defining set functions $f_1(S)$, $f_2(S)$, $f_3(S)$. Is $f_2$ a matroid rank function? $f_3$?

   **Question:** Give an interpretation of $f_3$.

2. (**Federico Ardila**) $T_{K_n}(1, -1) = A_{n-1}$, the number of alternating i.e. up-down permutations of $n-1$. The only proof I know is computing generating functions of both sides.

   **Problem:** Give a better explanation.

3. (**Oliver Lorscheid**) What is the relationship between CSM-balancing (in the sense of Lopez de Medrano et. al.) and higher balancing (in the sense of Lorscheid et. al.)?

### 3.8 Weak and strong maps

1. (**Alex Fink**) One might naturally ask, if $M \to N$ is a weak map of connected matroids of the same rank on the same ground set (i.e., every basis of $N$ is a basis of $M$), if there exist a regular matroid polytope subdivision of $M$ of which $N$ is a face. The answer, as shown in a paper of Brandt-Speyer, is in general **no**.

   **Question**: Can we salvage this by merely asking for a chain of subdivisions

$$M = M_0 \to M_1 \to \cdots \to M_k = N?$$

2. **(June Huh)** Can one prove that certain matroid polynomials (e.g. the Kazhdan–Lusztig polynomial and the Z-polynomial) are (coefficientwise) monotone with respect to weak maps?

3. **(Alex Fink)** Can we model weak maps using matroids over bands, in the sense of Baker–Bowler and Baker–Lorscheid?

   (Alex proposed a specific band which should do the job.)

4. **(Oliver Lorscheid)** If $f : M \to N$ is a strong map of matroids on the same ground set (i.e., $N$ is a quotient of $M$), there is a factorization theorem which says that $f$ factors as a restriction followed by a contraction. One might wonder if this also holds for $B$-matroids, where $B$ is an *idyll*. However, the answer is **no**: for example, when $B$ is the sign hyperfield, we are talking about oriented matroids and Richter-Gebert has given a counterexample to the factorization theorem.

   **Question** For which $B$ is it true?

# 4 Scientific Progress Made

Here we survey some of the preprints and collaborations that have already resulted from the 5-day workshop. The BIRS workshop was an excellent meeting ground for established collaborations. For example, Ardila, Dehnam, Eur, and Huh were able to establish a new project combining the stellahedral and the conormal perspectives on matroids. Alex Fink, Kris Shaw, and David Speyer were able to discuss work from a previous Fields institute semester, on the reduction of positivity of Speyer's g-polynomial. Omid Amini, Emanuele Delucchi, Alex Fink, Diane Maclagan, and Nolan Schock are collaborating on Chow rings of compactifications of divisorial arrangement complements. During the workshop, Rudi Pendavingh, Johannes Rau, Chi Ho Yuen were able to finish their ongoing project on amoeba dimensions [11] since three of the authors were present at BIRS. Dave Jensen and Sam Payne had an important breakthrough and were able to update a previous preprint to include a proof that the tropicalization of a linear series on an algebraic curve is finitely generated as a tropical module [15]. This was done using a key fact about valuated matroids.

Following a discussion during a problem session, Matt Baker and Oliver Lorscheid have provided a proof of a theorem often attributed to Lafforgue but missing from the literature [8].

Following conjectures on the vanishing of cohomology of tautological bundles presented by Matt Larson, Andrew Berget, Chris Eur, and Alex Fink have begun working on the analogue of the matroid Schubert variety for pairs of matroids to tackle this problem. Chris Eur's preprint on the cohomology of tautological bundles of matroids also follows up from these problem sessions [12]. Eur and Larson are also working on a project on K-theoretic positivity for matroids.

Matt Baker, June Huh and Oliver Lorscheid discussed applications of F1-geometry to unimodality of the number of matroids of diverse ranks on a fixed ground set. Like many projects mentioned here, this on-going collaboration will most likely pick up momentum during the special year on matroid theory at the IAS in Princeton in 2024.

Many new collaborations were also formed. For example, Nick Proudfoot reported that he also mentioned that, thanks to the hybrid format, Luis Ferroni was able to watch his talk online. This led to very interesting conversations, new conjectures, and counter-examples to old conjectures. Benjamin Schroeter reported that this presentation also led to discussions with Proudfoot and Lorenzo Vecchi, and new work on valuative invariants of (fundamental) transversal matroids. Matt Larson reported that, thanks to conversations during the workshop with Nima Anari and Chris Eur, he was able to obtain some non-trivial inequalities on the number of independent sets of certain delta-matroids, resulting in the preprint [16].

Many participants reported on new collaborations formed between mathematicians in different career stages. For example, Matt Baker and Oliver Lorscheid reported on collaborations with Tong Jin and Zach Walsh. Tong Jin also reported on helpful conversations on orthogonal matroids with Nathan Bowler.

# 5 Outcome of the Meeting

The overall feedback from the meetings participants was overwhelmingly positive. On top of the scheduled talks and discussions, the in-person event at BIRS' spectacular location provided many occasions for informal discussions and extra activities.

For many participants, it was one of the first time since before the COVID-19 pandemic that they were among such a large group of researchers in this field. Very importantly, this workshop provided a meeting ground for mathematicians in a a wide cross-section of career stages. Many senior participants expressed the positive outcome of meeting the new up and coming researchers in the field. Moreover, many junior participants expressed gratitude for the opportunity to give talks at the meeting.

There were many invited participants who could not attend in person and expressed interest in participating remotely. However, we unfortunately experienced very low online attendance during the talks and even lower online participation in the discussion sessions.

Another unfortunate event, were participants being forced to cancel their travel due to Canadian visa procedures. Invited participant Manoel Jarra and organiser Felipe Rincón were not able to attend the meeting in person due to visa wait times. This problem disproportionately effects mathematicians who are from underrepresented groups. Though we understand that visa processing times are out of BIRS' control, sending out workshop invitations early and informing future organisers and participants of visa requirements and processing times could help prevent this unfortunate occurrence in the future.

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