Summary of problem sessions at BIRS workshop 23w5006: “Spinorial and Octonionic Aspects of G_2 and $\mathrm{Spin}(7)$ Geometry”

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We collect here various questions and ideas for future research as discussed during the Problem Sessions of the workshop "Spinorial and Octonionic Aspects of G_2 and $\mathrm{Spin}(7)$ Geometry".

1. **(Jason Lotay)** Recall the notion of “triality" from John Huerta's talk. We know about triality at the algebraic level: it is a symmetry between vectors and spinors for the normed division algebra $\mathbb{O}$. Suppose we have (M^8, Φ) which is an 8-dimensional manifold with a torsion-free $\mathrm{Spin}(7)$-structure Φ. Is there any geometric meaning (as opposed to the purely algebraic structure at the level of Clifford algebras and $\mathbb{O}$) of triality in this setting?

   Does it make sense to define the notion of “mirror triality" for a triple of objects similar to the notion of mirror manifolds and mirror symmetry?

   Some ideas related to the first question were suggested by **Gavin Ball:** If we look at $Gr_{\text{Cayley}}(4, 8)$, the Grassmannian of Cayley 4-planes in $\mathbb{R}^8$ and the Grassmannian of 3-planes in $\mathbb{R}^7$ then $Gr_{\text{Cayley}}(4, 8) \cong Gr(3, 7)$.

   If we look at the space of curvature tensors of a Spin(7)-manifold, i.e., (M^8, Φ) with a torsion-free $\mathrm{Spin}(7)$-structure Φ then that as an irreducible $\mathrm{Spin}(7)$-representation is isomorphic to $V_{0,2,0}$ which is also isomorphic to the space of curvature tensors of Ricci-flat 7-manifolds. This might give a hint for the geometric implication of triality for $\mathrm{Spin}(7)$-manifolds.

   In fact, an analogous question would be that if we have two $\mathrm{Spin}(7)$-manifolds M^8_+, M^8_- then do they relate to a Ricci-flat 7-manifold if we have a geometric notion of triality?

2. **(Spiro Karigiannis)** If b^2, b^3 denote the 2nd and 3rd Betti numbers of a G_2-manifold then $b^2 + b^3$ is invariant under “mirror symmetry" for G_2-manifolds, i.e, they remain the same for the mirror manifolds. There is a notion of conifold transition in Calabi–Yau geometry and an analogous idea of G_2 conifold transitions has been given by Atiyah–Witten [**Atiyah-Witten**]. Recall that a G_2-manifold M^7 is called **semi-flat** if M is a coassociative fibration and the fibers are flat tori T^4. What can be said about the G_2 conifold transitions in the semi-flat case?

3. **(Jesse Madnick)** Construct non-trivial compact associative submanifolds in the Aloff–Wallach spaces $N_{k,l}$ with $(k, l \neq (1, 1))$ where $N_{k,l} = \dfrac{\mathrm{SU}(3)}{\mathrm{U}(1)_{k,l}}$ with its homogeneous nearly parallel G_2-structure.

What can be said about the conformal structure of associatives $\Sigma^3 \subset M^7$? An idea would be to use harmonic spinors just like one uses holomorphic sections to study conformal structures for holomorphic curves.

Can an **open** Riemann surface be conformally embedded in S^6? It's a theorem due to Robert Bryant that closed Riemann surfaces can be conformally embedded in S^6.

4. **(Sergey Grigorian, Spiro Karigiannis, John Huerta.)** Consider **gerbes** on G_2-manifolds. Suppose we have a manifold with a closed G_2-structure, i.e., (M^7, φ) with $d\varphi = 0$ and $[\varphi] \in H^3(M, \mathbb{Z})$. Is there any relation between $\mathrm{U}(1)$-gerbes on (M^7, φ) and $[\varphi]$. Or, consider $d * \varphi = 0$ and $[*\varphi] \in H^4(M, \mathbb{Z})$. Does there exist a relation between a 2-gerbe over M^7 and $[*\varphi]$?

   A motivation to study these questions come from Kähler geometry and to try to come up with a "G_2-Calabi–Yau theorem". A more precise but still vague question is the following: Recall that if we have a Kähler manifold (M^{2m}, g, J, ω) and we consider the canonical bundle $K = \Lambda^{m,0}(T^*M)$, then it is a line bundle over M, and Yau's proof of the Calabi conjecture states that $c_1(K) = 0 \iff$ there exists a Ricci-flat metric with its Ricci form in $[\omega]$. Here $c_1(K)$ is the first Chern class of K.

   So now suppose we have a manifold with a G_2-structure (M^7, φ, g). Does there exist some "canonical gerbe K" on M such that $c_1(K) = 0 \in H^3(M, \mathbb{Z}) = 0 \iff$ there exist a torsion-free G_2-structure in $[\varphi]$? Here $c_1(K)$ is the "first Chern class of the gerbe K" or more precisely the Dixmier–Douady class. Note that the question as stated is particularly vague because we still do not understand the actual notion of gerbes on G_2-manifolds and their relation to the torsion-freeness of the G_2-structure.

   More information about gerbes can be found in [**GO**; **Hitchin**; **MM**].

5. **(Mario Garcia-Fernandez)** Consider the heterotic G_2-system: that is, we have a compact (M^7, φ), $P \to M$ is a principle G-bundle with compact G and A a connection on P, and let $\alpha' \in \mathbb{R}$ be such that $dH = \alpha' \langle F_A \wedge F_A \rangle$, where $H \in \Omega^3(M)$.

   (a) Is there a spinorial interpretation for the cases when the torsion component $\tau_0 \neq 0$?

   (b) There have been both exact and approximate solutions of the heterotic G_2-system. Construct large classes of solutions and maybe solutions with large volume?

   (c) Is there a geometric flow to study the heterotic G_2-system?

   (d) From considerations in physics, α' is hoped to be "small". Do the solutions proposed in Leander Stecker's talk in conference (based on his work with Mateo Galdeano) have small α'?

   (e) Consider a sequence of heterotic G_2-systems $\{(M_{\alpha'_n}, \varphi_{\alpha'_n}, A_{\alpha'_n})$ where α'_n is a sequence in $\mathbb{R}$ and suppose that $\alpha'_n \to 0$. What can be said about the limit?

   (f) In the case of part (e), suppose that $M_{\alpha'_n} = M_{\alpha'} = M$ and that it admits a torsion-free G_2-structure φ. Do we have $\varphi_{\alpha'_n} \to \varphi$ and $A_{\alpha'_n} \to A$ with A a G_2-instanton?

   (g) If $M_{\alpha'}$ does not have a torsion-free limit then what happens to the limit? Does the limit collapse? Is the limit a soliton? (There is a notion of a heterotic G_2-system being a soliton)

   (h) Can we construct solutions of the heterotic G_2-system with $\alpha' \neq 0$ from a limit?

6. **(Henrique Sà Earp)** Consider instantons of Sasakian 7-manifolds, i.e., we take $\sigma \in \Omega^3(M)$ with $\sigma = \eta \wedge d\eta$ with η the contact 1-form to define the notion of instanton. For the Sasakian

case (which could be viewed as transverse-Kähler geometry) we have $d\eta = \omega$ and hence $\sigma = \eta \wedge \omega$. The space of 2-forms decompose further with

$$\Omega^2 = \Omega^2_V \oplus \Omega^2_H$$

and furthermore

$$\Omega^2_H = \Omega^2_8 \oplus \Omega^2_6 \oplus \Omega^2_1.$$

Instantons A with $F_A \in \Omega^2_8$ are self-dual contact instantons. If we look at the moduli space of self-dual contact instantons $\mathcal{M}_{SDCI}$ then one can show that dim $\mathcal{M}_{SDCI} =$ ind $\not{D}$ and $\mathcal{M}_{SDCI}$ is Kähler on its smooth locus.

(a) Can we define an orientation on $\mathcal{M}_{SDCI}$?
(b) What happens to the blow-ups, bubbling, and compactifications of $\mathcal{M}_{SDCI}$?
(c) Suppose we consider the 3-Sasakian case. Can we prove that $\mathcal{M}_{SDCI}$ is hyperKähler?

7. **(Jesse Madnick)** Can we say something about the non-zero torsion classes of a G_2-structure, the appearance of which will be a necessary and sufficient condition for every G_2-instanton being a Yang–Mills connection?

8. **(Gonçalo Oliveira)** There is a result of Derdzinsky from the 80s which says that "(M^4, g, ω) extremal (i.e., ∇S is a holomorphic vector field with S the scalar curvature) and g Bach-flat $\implies (M^4, S^{-2}g)$ is Einstein."

   Can we find conditions on (N^5, g, η, Φ) which is a Sasakian 5-manifold and is extremal, analogous to Bach-flatness in the 4-dimensional case, which would imply the existence of a conformal metric which is Einstein?

9. **(Spiro Karigiannis)** Let α be a calibration k-form on $\mathbb{R}^n$ equipped with the standard metric and orientation. (That is, α has constant coefficients and comass one.) Let $G = \mathrm{Stab}_{\mathrm{O}(n)}\alpha$. There are several properties that α may or may not have. These are the following:

   (a) G acts transitively on the unit sphere S^{n-1} in $\mathbb{R}^n$.
   (b) G acts transitively on the Stiefel manifold $V_{r,n}$ of r-tuples of orthonormal vectors in $\mathbb{R}^n$ for $1 \leq r \leq k-1$. (Note that (a) is just (b) for $r = 1$.)
   (c) G acts transitively on the Grassmanian Gr_α of α-calibrated k-planes in $\mathbb{R}^n$
   (d) Let W be an α-calibrated k-plane in $\mathbb{R}^n$, and let $H = \{P \in G : P(W) = W\}$ be the stabilizer in G of W. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G, H respectively. Then we can write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp_{\mathfrak{g}}}$. We always have the equality $\mathfrak{h} = \Lambda^2(W) \oplus \Lambda^2(W^\perp)$ and the inclusion $\mathfrak{h} \supseteq \mathfrak{g} \cap (W \otimes W^\perp)$. Property (d) is that the inclusion is an equality. We say such an α is *compliant*. If property (c) holds, then property (d) is independent of the choice of $W \in \mathrm{Gr}_\alpha$.
   (e) Suppose that (c) holds. Let $W \in \mathrm{Gr}_\alpha$. Let $e_1, \ldots, e_k$ be an oriented orthonormal basis of W and let $\nu_1, \ldots, \nu_{n-k}$ be an oriented orthonormal basis of $W^\perp$. Then in terms of the decomoposition $\Lambda^k(\mathbb{R}^n) = \Lambda^k(W \oplus W^\perp) = \oplus^k_{p+q} \Lambda^p(W) \otimes \Lambda^q(W^\perp)$, we can write $\alpha = \sum_{p+q=k} \alpha_{p,q}$. Property (e) is that only even values of q occur in this decomposition. One can show that (e) implies (d).
   (f) For any $v \in S^{n-1}$, both $v \lrcorner \alpha$ and $v \lrcorner \star \alpha$ have comass one. This is equivalent to the fact that any unit vector v lies in an α-calibrated k-plane and also lies in a $(\star\alpha)$-calibrated $(n-k)$-plane. We say that such an α is *rich*.

(g) For $v \in S^{n-1}$, let $L_v = \mathrm{Span}\{v\}$, so $\mathbb{R}^n = L_v \oplus L_v^\perp$. Write $\alpha = v \wedge \beta_v + \gamma_v$ where $v \lrcorner \beta_v = v \lrcorner \gamma_v = 0$, so $\beta_v \in \Lambda^{k-1}(L_v^\perp)$ and $\gamma_v \in \Lambda^k(L_v^\perp)$. Property (g) is that $\langle \beta_v, w \lrcorner \gamma_v \rangle = 0$ for all $w \in \mathbb{R}^n$ and all $v \in S^{n-1}$.

For some mysterious reason, every single one of the above properties is satisfied by the interesting geometric calibration forms (Kähler, special Lagrangian, associative, coassociative, Cayley). Several of these properties are in some sense quantifying that there are many α-calibrated k-planes. Is there a single property that a calibration α could have which implies all of these? If so, what is the geometric significance of such a property?

10. **(Daniel Platt)** Let $s : \mathbb{T}^3 \to X^{4k}$ (hyperKähler) and let $\{x_1, x_2, x_3\}$ be coordinates on $\mathbb{T}^3$ and I_1, I_2, I_3 be the triple of complex structures on X^{4k}. The Fueter operator on s is $Fs = \sum_{i=1}^3 I_i \left(ds(\frac{\partial}{\partial x_i}) \right)$. If $Fs = 0$ then s is called a Fueter section. A known fact about F is that it is an index 0 operator and hence the expectation is that Fueter sections are rigid. However, all know examples of X^{4k} where s is explicit have moduli. So can we have Fueter sections which do not have moduli, i.e., that are rigid?

11. **(Jason Lotay)** Suppose $E \to (M^7, \varphi)$ with M being a G_2-manifold. Suppose A is a G_2-instanton on E. What does it tell us about E? The situation we have in mind is that of bundles over Kähler manifolds, where existence of Hermitian–Yang–Mills connection $\iff$ the bundle is stable due to Donaldson–Uhlenbeck–Yau theorem or the Kobayashi–Hitchin correspondence. So the questions are 1) Is there a notion of stability of bundles E over a G_2-manifold? 2) Is there a Donaldson–Uhlenbeck–Yau/Kobayashi–Hitchin correspondence type theorem?

    The loop space of a G_2 manifold is a Calabi–Yau manifold. Can we use this information and point of view to get a notion of stability and answer above questions?

    Is there a Geometric Invariant theory, moment map and/or symplectic reduction picture associated with the not-yet-defined notion of stability?

12. **(Spiro Karigiannis)** In the Kähler case, $\Omega^2 = \Omega^{2,0} \oplus \Omega^{0,2} \oplus C^\infty \omega \oplus \Omega_0^{1,1}$ and a connection A is Hermitian–Yang–Mills $\iff$ $F_A \in \Omega_0^{1,1}$. Now consider a gerbe over (M^7, φ) and let A be a connection on the gerbe. Then F_A is a 3-form on M. Is it true that F_A is "Hermitian–Yang–Mills" $\iff$ $F_A \in \Omega^3_{27}$?

13. **(Gavin Ball)** Consider the standard G_2-structure φ on $\mathbb{R}^7$ and let S be the set of degenerate 3-forms on $\mathbb{R}^7$. S is singular. What can we say about dist(φ, S), the distance between φ and S? We know that dist$(\varphi, S) \leq 1$ but can it be smaller?

References

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