Inclusive Paths in Explicit Number Theory Summer School

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1 Overview of the Field

The subject area of this event was analytic number theory, which focuses on arithmetic questions through the lens of L-functions. While there has been extensive theoretical work done concerning L-functions and estimates for prime-counting functions, these results have usually been inexplicit in nature. In recent years there has been a plethora of important results in explicit number theory, including the resolution of the Odd Goldbach Conjecture by Helfgott and Platt [12, 13] and prior to that Tao's improvement [39] of Ramaré's work on Schnirelmann's constant [35]. All of these works use, in an essential way, explicit results about primes and about zeros of L-functions.

2 Recent Developments and Open Problems

Since 2015, there have been a striking number of results about sharp bounds for prime counting functions, improving some first results of Rosser and Schoenfeld [37] which had been referenced over 1000 times (see [2, 5, 6, 9, 10, 33]). These results were prompted by a combination of extensive computational work, widened zero-free regions, and the first explicit zero-density results for the Riemann zeta function, with contributions of Kadiri, Ng, Platt, Ramaré, Trudgian [20, 21, 31, 36, 43] and their students and postdocs (see [5, 7, 8, 10, 11, 24]). There has also been a lot of progress on explicit subconvexity results with contributions of Hiary, Patel, Platt, Trudigan, and Yang [14, 15, 29, 32] and on large-scale computations of zeros of the Riemann zeta function by Bober, Hiary, Platt, and Trudgian [4, 31, 34]. Questions on primes in more general contexts are increasingly harder to answer, although Bennett, Martin, et al. [3] proved in 2018 some extensive results for primes in arithmetic progressions using new large-scale partial verifications of RH and GRH (Platt's [31]) and explicit results for the zeros of Dirichlet L-functions. Since 2016, Kadiri, Ng, Lumley, and Ramaré [19, 21, 36] have determined explicit regions where Dirichlet L-functions do not vanish (zero-free regions) and given estimates for the count of their zeros off the critical line (zero-density results). Zero-density is a topic that has been somewhat ignored in recent years. Despite this there have been recent important applications of zero-density results to the least prime in number fields, Chebotarev's Density Theorem [21, 40, 41], bounds for ℓ-torsion of class groups [26, 30], moments of the zeta function [28], and primes in arithmetic progressions [42]. These resources have already been cited over 400 times and are useful for a wide array of problems, including Diophantine equations such as generalized Fermat equations and bounds for the first prime in the Chebotarev density theorem [24]. Since the results of Kadiri and Ng [22] on Dedekind zeta functions, important explorations of explicit number theory in the context of modular forms and the least prime in the Chebotarev density theorem have sprouted, many due to Ono's REU research groups [17] as well as to Kadiri, Ng & Wong, Thorner, Zaman, and Ahn & Kwon [1, 22, 23, 19, 40]. The advances have now reached the point where concerted research efforts have the potential to achieve significantly stronger bounds for the least prime in this more general setting.

3 Scientific Activities and Presentation Highlights

This summer school brought together many of the leading experts in the field of Explicit Number Theory for the first time. This event gave students the unique opportunity to be immersed in this field and to learn some of the key topics very rapidly, through 11 lectures and several learning activities that focused on numerical explorations and computational methods. The courses were consciously designed to be interconnected, and they covered three general topics:

1. Zero-free regions and zero-repulsion of zeros of L-functions and the Burgess inequalities;
2. Subconvexity, zero-density results and applications;
3. Chebotarev's density theorem.

The topics were tailored to prepare the students for the research projects that took place in the second week of the summer school. A number of the topics on this list are very advanced and they are not typically taught in university graduate courses. One of the unique features of the summer school was learning activities associated to each lecture. After each lecture, students worked together in groups of five on problems and exercises carefully selected by the course lecturers.

Among the speakers, Kadiri, Treviño, and Wong covered topic 1, Hiary, Ramaré, Vatwani, and Zaman covered topic 2, and Fiori, Hamieh, Murty, and Sinha covered topic 3. In addition to these three general topics, Trudgian gave a very nice overview of the key goals and some of the main topics in explicit number theory on the first day of the summer school.

In topic 1, Kadiri gave a thorough exposé of zero-free regions by providing original insight on how to explore low-lying zeros, the use of trigonometric inequalities, and Stechkin's trick. Wong continued on this topic by giving an exposition of the Deuring–Heilbronn phenomenon describing zero repulsion for Dedekind zeta-functions, as well as the details of the power sum method. Finally, Treviño gave a very clear and detailed overview of the proof of Burgess's famous and difficult bound for character sums, which is a central tool for studying L-functions and establishing zero-free regions.

In topic 2, Hiary gave an overview of the state of the art in subconvexity, explaining carefully the A and B processes and listing many of the key results. Ramaré gave a lecture on the general method of computing zero-density results for L-functions. Zaman's lecture on how to compute log-free zero-density estimates using the mollifier method was original and complemented nicely Ramaré's presentation. Zaman also explained the connection between log-free zero-density estimates and Linnik's theorem. Vatwani closed the topic by presenting a clear and detailed account of how zero-density results could be applied effectively to computing primes in short intervals.

In topic 3, Sinha gave a concise but complete general overview of the main results in a first algebraic number theory course. This provided participants with the background material required for understanding Chebotarev's density theorem. Hamieh presented the seminal work of Lagarias and Odlyzko on the Chebotarev density theorem by giving a complete proof of the effective version of the theorem. Fiori's learning activity complemented Hamieh's talk with an extensive list of theoretical and computational problems related to Chebotarev's density theorem and to the least prime in Chebotarev's density theorem. Murty concluded Week 1 with an insightful lecture on applications of Chebotarev's density theorem including work on Artin's primitive root conjecture.

In the second week of the summer school the summer school, participants were divided into ten research groups of 4–6 people, each led by one of the organizers or speakers. The projects are described in full detail in Section 5 below. The projects aimed at establishing explicit versions of many fundamental results in analytic number theory such as the prime number theorem, Linnik's theorem, the Deuring–Heilbronn phenomenon, Burgess's inequality, subconvexity bounds for $\zeta(s)$, and zero-free regions for various families of L-functions. By the end of the second week many of these projects made significant advances and in some cases theorems were established. All of the projects had the goal of being completed and submitted to a scholarly journal within nine months of the end of the summer school. This research output represents significant and accelerated advances in the field.

4 Speakers, Titles and Abstracts

4.1 Lectures and Learning Activities

The first week of IPENT featured mini-courses delivered in a hybrid format. With the exception of Trudgian's and Murty's lectures, all the other lectures were followed by a 60- to 90-minute problem solving sessions. In what follows we list the minicourses in chronological order.

Zero-free regions close to the real axis by Habiba Kadiri (University of Lethbridge)
Course Description: The zero-free regions for the Riemann zeta function uses bounds for zeta at large enough heights. This is not feasible when we look at the case of other Dirichlet L-functions or Dedekind zeta functions. Instead, we must consider the existence of low-lying zeros, real ones, and even of an exceptional one close to 1. In this lecture, we will assume the audience to be familiar with classical proof for zero-free regions for zeta, and will focus on the techniques to establish regions close to the real line with at most one (or even a finite number) of zeros.

Not another improvement in some random constant . . . by Tim Trudgian (UNSW Canberra)
Course Description: What's the point in going through a proof and changing $O(x)$ to something of size at most $20x$? Isn't this just bookkeeping? Why bother spending another paper improving it to $10x$? These are the sorts of questions one may be asked when conducting work on explicit estimates in number theory. Out of pure self-interest if nothing else, I will outline reasons why one should pursue such research.

An explicit Burgess inequality by Enrique Treviño (Lake Forest College)
Course Description: We will prove an explicit version of the Burgess inequality that bounds short character sums. If time permits, we will apply the inequality to give a bound on the least quadratic non-residue.

Review of algebraic number theory by Kaneenika Sinha (IISER PUNE)
Course Description: We will review basic principles in algebraic number theory that are needed to understand Artin L-functions and the Chebotarev Density Theorem. The goal of these lectures is to provide the necessary background for the talks of A. Hamieh and R. Murty.

The Riemann zeta function, explicit bounds, and exponent pairs by Ghaith Hiary (Ohio State University)
Course Description: In lecture 1, we discuss explicit bounds for Riemann zeta function. In lecture 2, we discuss explicit bounds for Dirichlet L-function.

Zero-free regions of Dedekind zeta functions and the Deuring–Heilbronn phenomenon by Peng-Jie Wong (National Sun Yat-Sen University)
Course Description: Similar to Dirichlet L-functions, each Dedekind zeta function $\zeta_L(s)$ of a number field L admits a "standard" zero-free region with a possible exceptional zero. The Deuring–Heilbronn phenomenon roughly asserts that if the exceptional zero exists, such a standard zero-free region for $\zeta_L(s)$ can be enlarged. A precise version of the Deuring–Heilbronn phenomenon for Dedekind zeta functions was obtained by Lagarias, Montgomery, and Odlyzko, who used Turán's power sum method. Their work was made explicit by Zaman (for d_L sufficiently large) and Ahn–Kwon (for all number fields). In addition, with Kadiri and Ng, we appeal to Harnack's inequality to bring some improvements to these results. In this lecture, we shall discuss how to apply Harnack's inequality to improve Turán's power sum method introduced by Lagarias, Montgomery, and Odlyzko, and then we will sketch the proof of the Deuring–Heilbronn phenomenon for Dedekind zeta functions. We will mainly follow the joint work with Kadiri and Ng (*The least prime ideal in the Chebotarev density theorem*, Proc. Amer. Math. Soc. 147 (2019), 2289–2303).

A quick introduction to density estimates by Olivier Ramaré (CNRS and Institut de Mathématiques de Marseille)

Course Description: In lecture 1, we cover some historical pointers and zero detection methods. In lesson 2, we cover density estimates—usage of moments, and sketch of a full estimate.

An introduction to log-free zero density estimates by Asif Zaman (University of Toronto)
Course Description: Linnik proved his breakthrough result on the least prime in an arithmetic progression in 1944, and one of his pivotal innovations was a log-free zero density estimate. In this summer school lecture, I will introduce log-free zero density estimates, review some of their history and applications, and explore two essential questions. First, why does "log-free" matter? Second, how can you prove this kind of result?

Short intervals containing primes by Akshaa Vatwani (IIT Gandhinagar)
Course Description: In 1930, Hoheisel proved that there exists a $\theta < 1$ such that the interval $(x, x + x^{\theta+\varepsilon})$ contains a prime for $x > x_\theta(\varepsilon)$. His proof relies on two ingredients: a zero-free region for $\zeta(s)$, and a zero density estimate for $\zeta(s)$, that is, an estimate for $N(\sigma, T)$ which is the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\beta \geq \sigma$ and $\theta \leq \gamma \leq T$. Ingham (see also Ch. 14 of Montgomery's *Topics in multiplicative number theory*) showed an explicit relationship between the estimate for $N(\sigma, T)$ and the value of the exponent θ above. In this lecture, we will discuss the proof of this result. We will also mention related applications, such as primes between consecutive powers (see D. Bazzanella, *Primes between consecutive powers*, Rocky Mountain J. Math. 39 (2009), no. 2, 413–421).

Artin L-functions and the proof of the Chebotarev Density Theorem by Alia Hamieh (University of Northern British Columbia)
Course Description: We study the analytic properties of Artin L-functions and Hecke L-functions that are needed in the proof of the Chebotarev density theorem. We demonstrate how this theorem generalizes many classical results on the distribution of primes and prime ideals. For example, we describe the connection between the Chebotarev density theorem and Dirichlet's theorem for primes in arithmetic progressions. We sketch the proof of the classical version of Chebotarev density theorem (without an error term) following Chebotarev's original treatment. We then present the work of Lagarias and Odlyzko on an effective version of Chebotarev density theorem.
Learning Activities: Andrew Fiori prepared and delivered a 90-minute tutorial that featured an extensive list of theoretical and computational problems focusing on numerical exploration of the Chebotarev's density theorem and the least prime in Chebotarev's density theorem.

Applications of the Chebotarev's Density Theorem by Ram Murty (Queen's University)
Course Description: Using the background material covered by Professors Hamieh and Sinha, I will give survey of some of the applications of the Chebotarev density theorem.

4.2 EDI sessions

We hosted two EDI sessions during the summer school. The sessions were led by Keira Gunn who is a PhD student in mathematics at the University of Calgary. She engaged in discussions about the necessity of EDI in mathematics and about the challenges of being a junior mathematician, especially when coming from a historically marginalized group. Participants were activelly involved via group discussions, homeworks, and surveys.

EDI: Why does it matter?, facilitated by Keira Gunn
Description: In this first two-hour session, we will explore the meanings of the terms equity, diversity, and inclusion (EDI), and how they are relevant to a discipline such as mathematics which might (on first glance) seem "beyond bias". We will examine real-world biases in real-world "objective" systems and look at how the experience of female scientists differs from that of male scientists. Through reflection on one another's experiences, we will be able to articulate why EDI matters to mathematicians.
Take-home written project: What have you done, or what can you see yourself doing to make mathematics a more inclusive and equitable field? Write a 500+ word statement or equivalent presentation in another format.

Challenges as a Budding Mathematician: Changing the narrative, facilitated by Keira Gunn
Description: After sharing our thinking about EDI since last week's session, we will discuss in this second two-hour session studies of inequities in career-related processes such as instructor evaluations. We will take part in an exercise designed to illustrate the notion of privilege and to recognize the privilege (or lack

thereof) that we experience ourselves. Finally, we will discuss the notation of allyship and how we can use our influence for positive change in the discipline of mathematics.

4.3 Poster Session

We started Week 2 of IPENT with a poster session featuring the following posters.

Sign changes in the error term of the Piltz divisor problem, by Cruz Castillo (University of Illinois at Urbana–Champaign)
Abstract: For an integer $k \geq 3$, let $\Delta_k(x) := \sum_{n \leq x} d_k(n) - \mathrm{Res}_{s=1}(\zeta^k(s)x^s/s)$, where $d_k(n)$ is the k-fold divisor function, and $\zeta(s)$ is the Riemann zeta-function. In the 1950's, Tong showed for all large enough X, $\Delta_k(x)$ changes sign at least once in the interval $[X, X + C_k X^{1-1/k}]$ for some positive constant C_k. For a large parameter X, we show that if the Lindelöf hypothesis is true, then there exist many disjoint subintervals of $[X, 2X]$, each of length $X^{1-\frac{1}{k}-\varepsilon}$, such that $\Delta_k(x)$ does not change sign in any of these subintervals. If the Riemann hypothesis is true, then we can improve the length of the subintervals to $\gg X^{1-\frac{1}{k}}(\log X)^{-k^2-2}$. These results may be viewed as higher-degree analogues of a theorem of Heath-Brown and Tsang, who studied the case $k = 2$. This is joint work with Siegfried Baluyot.

Hardy–Littlewood 3-tuple prime conjecture, by Shivani Goel (IIIT Delhi, India)
Abstract: The Hardy and Littlewood k-tuple prime conjecture is one of the most enduring unsolved problems in mathematics. In 1999, Gadiyar and Padma presented a heuristic derivation of the 2-tuples conjecture by employing the orthogonality principle of Ramanujan sums. Building upon their work, we explore triple convolution Ramanujan sums and use this approach to provide a heuristic derivation of the Hardy-Littlewood conjecture concerning prime 3-tuples. Furthermore, we estimate the triple convolution of the Jordan totient function using Ramanujan sums.

Counting zeros of zeta functions, by Elchin Hasanalizade (University of Lethbridge)
Abstract: Let $N(T)$ and $N_K(T)$ denote the number of non-trivial zeros up to height T of the Riemann zeta function $\zeta(s)$ and Dedekind zeta functions $\zeta_K(s)$, respectively. We obtain new explicit bounds for $N(T)$ and $N_K(T)$ which improve previous result of Kadiri and Ng, and Trudgian. The improvement is based on ideas from the recent work of Bennett et al. on counting zeros of Dirichlet L-functions. This is a joint work with Quanli Shen and Peng-Jie Wong.

Conditional estimates for logarithms and logarithmic derivatives in the Selberg class, by Neea Palojärvi (University of Helsinki, Finland)
Abstract: The Selberg class consists of functions sharing similar properties to the Riemann zeta function. The Riemann zeta function is one example of the functions in this class. The estimates for logarithms of Selberg class functions and their logarithmic derivatives are connected to, for example, primes in arithmetic progressions. In this poster, I will present effective and explicit estimates for logarithms and logarithmic derivatives of the Selberg class functions when $\Re(s) \geq 1/2 + \delta$ where $\delta > 0$. All results are under the Generalized Riemann hypothesis and some of them are also under assumption of a polynomial Euler product representation or the strong λ-conjecture. The poster is based on a joint work with Aleksander Simonič (University of New South Wales Canberra).

An induction principle for the Bombieri-Vinogradov theorem over $\mathbb{F}_q[t]$, by Aditi Savalia (Indian Institute of Technology Gandhinagar)
Abstract: The Bombieri-Vinogradov theorem establishes that the primes are equidistributed in arithmetic progressions "on average" for moduli q in the range $q \leq x^{1/2-\epsilon}$ for any $\epsilon > 0$. Let $\mathbb{F}_q[t]$ be the polynomial ring over the finite field $\mathbb{F}_q$. For arithmetic functions $\psi_1; \psi_2 : \mathbb{F}_q[t] \to \mathbb{C}$, we establish that if a Bombieri-Vinogradov type equidistribution result holds for ψ_1 and ψ_2, then it also holds for their Dirichlet convolution $\psi_1 * \psi_2$. As an application, we obtain an asymptotic for the average behavior of the divisor function over shifted products of two primes in $\mathbb{F}_q[t]$. This is joint work with Sampa Dey.

On the edge of the convolution method and a logarithmic sum related to the Selberg sieve, by Sebastian Zuniga-Alterman (University of Turku, Finland)
Abstract: We present a new method that improves qualitatively almost all instances of the convolution method under some regularity conditions; now, the asymptotic estimation of averages of well-behaved square-free

supported arithmetic functions can be given with its critical exponent and a reasonable explicit error constant. As an application, we study a logarithmic sum related to the Selberg sieve.

5 Scientific Progress Made

The second week of IPENT featured organized group research projects in explicit number theory led by senior researchers. All groups were successful in obtaining partial results toward their goals which they presented at the end of week 2 in a series of 10-minute talks. We aim to have the projects completed and submitted to a scholarly journal within nine months of the end of the summer school. This research output represents significant and accelerated advances in the field.

Bounds for the prime number theorem
Project team: Nizar Bou Ezz, Andrew Fiori (leader), Mikko Jaskari, and Sebastian Zuniga Alterman (co-leader)
Goals: This project aims to improve the best known bounds on the error term for $|\psi(x) - x|$, obtaining both results which hold for $x > x_0$ for some explicit x_0, and results which hold for all $x > 2$. We would also like to develop a clean software implementation of the methods that is adaptable to sums over zeros of other L-functions, and to convert the improved bounds on $|\psi(x) - x|$ to bounds on $|\theta(x) - x|$ and $|\pi(x) - \mathrm{Li}(x)|$. An ambitious goal would be to produce a version adaptable enough to handle more exotic "weights" (on the sum over zeros) as could arise for other explicit formulae.

Explicit zero-free regions for automorphic L-functions
Project team: Steven Creech, Alia Hamieh (leader), Kaneenika Sinha (co-leader), Jakob Streipel, Kin Ming Tsang, and Simran Khunger
Goals: This project aims to establish explicit zero-free regions for automorphic L-functions associated with degree-2 automorphic forms.

Bounding $\zeta(s)$ using numerical computation
Project team: Ghaith Hiary (leader), Neea Palojärvi (co-leader), Siva Nair, and Tianyu Zhao
Goals: The goal is to develop and implement an efficient numerical algorithm to bound the size of exponential sums of the form $\sum_{N \le n < N+L} n^{-\sigma - it}$ over a wide range of t. One application of this is to prove bounds on $|\zeta(1/2 + it)|$ in a new way that is, hopefully, less sensitive to mathematical errors.

Zero-free regions for Dirichlet characters of prime moduli
Project team: Abhay Chaudhary, Habiba Kadiri (leader), Nicol Leong (co-leader), and Subham Roy
Goals: The goal is to prove zero-free regions for Dirichlet L functions, and to make some result of Heath-Brown's explicit and valid for all prime moduli $q \ge 3$: there is an effectively computable constant R such that $\prod_{\chi \pmod q} L(s, \chi)$ has at most one zero in the region

$$\left\{ s = \sigma + it \colon \sigma \ge \frac{1}{R \log q},\ |t| \le 1 \right\}.$$

(Such a zero, if it exists, is real, simple, corresponds to a non-principal real character, and is labelled exceptional.) Such results are key for establishing explicit results about the primes (locating primes in short intervals, estimates for the error term in the prime number theorem in arithmetic progressions), as well as in applications to Diophantine problems. They are also useful for establishing an explicit version of Linnik's theorem about the size of the least prime in an arithmetic progression.

An explicit version of Linnik's theorem
Project team: Cruz Castillo, Michaela Cully-Hugill (co-leader), Jewel Mahajan, Nathan Ng (leader), and Chi Hoi (Kyle) Yip
Goals: This project aims to prove a completely explicit bound for the least prime in an arithmetic progression. Let $q \ge 2$ be a positive integer and $\gcd(a, q) = 1$. Dirichlet proved that there are infinitely many primes $p \equiv a \pmod q$. Let $P(a, q)$ denote the least prime $p \equiv a \pmod q$. Linnik proved that there exists $L > 0$ such that $P(a, q) \ll q^L$, for q sufficiently large. This result is often referred to as Linnik's theorem. The current published world record is $L = 5.2$, due to Xylouris. The goal of this project is to determine $C > 0$

and $L > 0$ such that $P(a,q) \leq Cq^L$ for all $q \geq 2$. That is, we shall remove the "q sufficiently large" condition.

Explicit result about sums of two primes in short intervals
Project team: Fatma Çiçek (co-leader), Marcella Manivel, Olivier Ramaré (leader), Alexander Slamen, and Hung-Liang Tsai
Goals: This project aims to prove that every interval of the shape $[x, x+x^{4/5}]$ contains a sum of two primes. Reducing the interval to $x^{3/5}$ will also be considered. In between, explicit results on the almost-everywhere distribution of the Tchebychef ψ-function will have to be proved, in an L^2-sense.

Explicit Burgess inequality for non-prime moduli
Project team: Elchin Hasanalizade, Hua Lin, Andradis Elieser Luna Martinez, Greg Martin (co-leader), and Enrique Treviño (leader)
Goals: Let χ be a Dirichlet character $\bmod q$. The Burgess inequality states that for q cubefree, for any integer $r \geq 2$ and any integers M, N (with $N > 0$), we have for any $\epsilon > 0$,

$$\left| \sum_{M<n\leq M+N} \chi(n) \right| \ll N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+\epsilon}.$$

Explicit bounds for the inequality are known if q is prime or if $r = 2$. This project aims to find explicit bounds for the Burgess inequality on composite moduli for $r \geq 3$.

The Hardy–Littlewood conjecture on $\pi(x+y)-\pi(x)$: bounds for y (proposed by Tim Trudgian)
Project team: Ertan Elma (co-leader), Nicolo Fellini, Akshaa Vatwani (leader), and Do Nhat Tan Vo
Goals: This project aims to improve the range of y for which the so-called Hardy–Littlewood conjecture $\pi(x+y)-\pi(x) \leq \pi(y)$ is true.

Zero-free regions for Dedekind zeta functions
Project team: Aditi Savalia, Sourabhashis Das, Swati Gaba, Ethan Lee (co-leader), and Peng-Jie Wong (leader)
Goals: The objective of this project is to improve the best known constants for the explicit zero-free region of Dedekind zeta functions $\zeta_L(s)$ established by Ahn and Kwon. In particular, we would like to adapt an argument of Kadiri to determine (and improve) the constants $A, B > 0$ such that $\zeta_L(s)$ has no zeros in the region

$$\Re(s) \geq 1 - \frac{1}{A \log d_L + Bn_L \log(|\Im(s)|+2)} \text{ and } |\Im(s)| \leq 1 \tag{1}$$

with the exception of at most one real zero β_1.

Zero repulsion estimate for Dirichlet characters
Project team: Kübra Benli (co-leader), Shivani Goel, Henry Twiss, and Asif Zaman (leader)
Goals: This project aims to establish a strong explicit version of zero repulsion (the "Deuring–Heilbronn phenomenon") for zeros $\rho = \beta + i\gamma$ of Dirichlet L-functions $L(s,\chi)$ which hold uniformly for any character $\chi \pmod q$ with $q \geq 3$ and any height $|\gamma| \leq T$. Ideally we want to nearly match the best known estimates for large values of qT. More ambitiously, we may attempt to achieve results comparable to more recent techniques which were applied for large modulus q and *fixed* height T.

6 Outcome of the Meeting

This summer school aimed to introduce students and junior researchers to various problems in explicit number theory. We provided training in the specifics of computational and explicit methods by having them work through problems involving primes and zeros of L-functions. We also facilitated opportunities for them to contribute to original research results, with an explicit goal of every participant being part of a research paper to be submitted within nine months. In addition, we organized two sessions on equity, diversity, and inclusion (EDI) and fostered inclusion implicitly throughout the summer school with icebreakers and networking activities.

The first week of the summer school was focused on lectures and learning activities in explicit number theory.

- The field of explicit number theory is well-suited for a summer school setting, as the bar for entry is not as high as in some other research fields. The event touched on several special topics about the zeros of L-functions, Chebotarev density theorem, and numerical investigations. Most participants did not begin with all this background but proved capable of acquiring it rapidly, together with facility in using computational software platforms. We had sent mathematical resources to the participants in the weeks leading to the summer school, which we found increased their active involvement in this first week.
- Explicit methods in the study L-functions have often been deemed afterthoughts to the more classical asymptotic results. This event gave participants the opportunity to focus on the specific challenges of explicit results, as well as to better comprehend the potential to extend techniques from the Riemann or Dirichlet setting to obtain new results about higher degree L-functions. In this way, the summer school contributed to bridging the gap between different fields of number theory. In addition, the courses had a specific focus on using numerical software. Mastery of tools such as Python, Sage, or Mathematica is a valuable asset for number theorists, and the summer school provided useful training in this perspective.
- The video recordings and the lecture notes for the Week 1 lectures can be found on the official BIRS website for this event.

The second week of the summer school was focused on research groups working towards specific publishable results in explicit number theory.

- We wanted to create a more systematic and organized approach to making progress in explicit number theory. With contributions from our senior participants, we created and shared a list of problems, which provided the basis for the ten collaborative research projects initiated at the event and still currently ongoing.
- We also wanted to encourage researchers to work more collaboratively. While many of the leading experts had occasionally collaborated (and even competed) with one another, this event was the first instance where they all gathered to discuss new directions and organize new collaborations for our field. The event appeared to greatly facilitate communication among these senior researchers, who welcomed the collaborative aspect of the event.
- Our aim was also to accelerate the production and quality of explicit results that require significant effort but are nevertheless within reach. While we expected participants to be invested in their groups' research and experts to be generous with sharing intersting projects, the results still exceeded our expectations. It was a very energizing experience, and we are looking forward to see the long-term effects of the summer school on research production.
- While the need for high-quality explicit results is high, this is still a marginalized area of research. This summer school has allowed to garner interest from young researchers, to educate them on advanced topics while giving them the opportunity to contribute to them with original results. As a consequence, we are optimistic about the long term goal of this school to contribute to the training to a new generation of mathematicians while at the same advancing knowledge in our field.

Throughout the summer school, we wove the principles of equity, diversity, and inclusion (EDI) into the planning and structure of the event.

- We started with the selection criteria for participants: through our application process, we learned about their mathematical interests and capability, their outlook towards collaborative work, their experience in a discipline in which many groups are unfairly underrepresented, and their commitment to upholding the inclusive values of our Code of Conduct.
- We scheduled multiple icebreakers, which were designed to create opportunities for junior and senior participants to interact with one another, as well as in-person and online participants. In these activities we consciously formed different groups of participants to maximize the diverse connections being made. We aimed to create a setting where people were comfortable being their true selves, and where participants built an environment that was mutually uplifting rather than competitive. The multiple strong and long-lasting relations that the school initiated indicated that participants had found a new supportive and creative community.

- We consciously included participants whose home institutions were smaller or did not have active number theory groups, to provide equal opportunity for learning about and actively participating in state of the art research. This principle guided our commitment to providing equal opportunities, ensuring that every student had the chance to enhance their research experience.
- Our goal was to facilitate mentorship for a new generation of more than 65 young researchers and to include them in our CRG network. The various group configurations created an ongoing mentoring network of faculty and postdocs to continue supporting the graduate students after the event. Moreover, we intentionally structured the research groups to include every in-person participant, to ensure that people from equity-deserving groups were on equal footing with people from majority groups who tend to have better access to collaboration.
- We scheduled discussions relating both to EDI generally and to our mathematical community specifically. Our two EDI sessions (one in each week) were led by an external facilitator who is at the same time a math graduate student themselves (thus someone most participants could relate to). They engaged in discussions about the necessity of EDI in mathematics and about the challenges of being a junior mathematician, especially when coming from a historically marginalized group. Participants were actively involved via group discussions, homework, and surveys. While the questions and issues raised were challenging at times, people seemed receptive and open to discussions. We found that the sessions helped challenge biases and raise awareness about differences in our experiences and opportunities.
- Another of our EDI goals was to provide an appreciation of the place where our event was hosted. We introduced our participants (the majority of them international) to the local scenery and culture via excursions that included a guided tour of the Indigenous art collection on campus. The artwork presented by the gallery curator and students gave an appreciation of the vibrant Indigenous culture while opening discussions around Reconciliation, as well as sexual violence in STEM (given that one art piece was a memorial to the Montréal Polytechnique massacre). Finally, the various excursions gave an appreciation of the spectacular natural beauty around Kelowna, which afterwards helped all of us identify strongly with the devastation to the UBC–O community caused by the recent wildfires.

7 Impact of the Hybrid Format

We have had positive experiences in the past with hybrid scientific events that were organized more like traditional conferences: online participants in distant time zones could simply choose for themselves which talks they were willing to attend. We wished to extend the IPENT content and community to participants unable to attend in person; however, in hindsight the summer school format was not ideal for including online participants. Because we were frequently splitting into small working groups for the learning activities, it was important to us that online participants attend for the full session each day; however, despite online participants pledging to attend for full days, few (less than 10) actually did so. As a result, planning the working groups was challenging since we couldn't predict the online presence in advance.

There was also an unfortunate circumstance where the computer network on the UBC–O campus had its functioning interrupted, and our main lecture room was unable to use its built-in computer to facilitate presenting the lectures to the online participants. We made do with personal laptops and tablets, but the ad hoc system was suboptimal for the online participants.

8 Demographics

We had 83 registered participants, among which 48 attended in person and 35 attended online.

- 21 of the 83 attendees were female, including 13 of the 48 in-person attendees.
- 15 attendees were professors, 7 were postdocs, and 61 were graduate students.
- 27 attendees were from Canada, 25 were from the USA, and 31 were from elsewhere.

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