Optimal transport for the next generation

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1 Overview of the Field

Optimal transport is an extremely popular and rapidly developing area of mathematics, with many diverse applications. Although first rate textbooks on this topic exist, including those by Villani [4, 5] and Santambrogio [3], they are now several years old and important topics have emerged since their publication, including, in particular, entropic regularization. Some of these topics are covered in the excellent books of Peyre-Cuturi [2] and Galichon [1]; however, these books focus on particular areas of application (computations in machine learning and economics, respectively). Our goal is to write a general and accessible textbook on optimal transport, covering the core theory, including the newer part, and a selection of important, more specialized topics. The target audience will be graduate students with strong quantitative backgrounds, in field such as mathematics, statistics, computer science and economics, with the main prerequisite being graduate level measure theory or probability theory.

2 Scientific Progress Made

Although one of us (Pal) was unable to attend the research in teams event in person due to a family emergency, he participated regularly via video conferencing, and we were still able to achieve our goal for the meeting. We hammered out a precise concept for our book: it will develop three disparate, but complementary viewpoints on optimal transport: the static and dynamic formulations of optimal transport, and stochastic aspects. In particular, the dynamic point of view, often an add-on in other presentations of optimal transport theory, will be introduced from the beginning and discussed along side its static counterpart. We believe this can be achieved, without sacrificing accessibility, by starting the book with an in-depth study of optimal transport of "empirical measures"; that is, measures uniformly supported on a fixed number of points. By working in this setting, core concepts of optimal transport can be developed while avoiding more subtle, technical functional analytic issues (these will of course need to be covered, but will be deferred until later chapters). We believe that in this setting, readers will be able to clearly see the emergence of the dynamical formulation of optimal transport as an extension from points to finitely supported measures of the dynamic formulation of distance on Euclidean space. We plan in this introductory chapter to fully develop the theory of optimal transport, Wasserstein distance and the Otto calculus on empirical measures. Stochasticity will be introduced at this stage as well, by considering the Schroedinger bridge between empirical measures, and noting how the static and dynamic formulations are recovered in the appropriate limit. Other stochastic and dynamic aspects of the field, including connections between entropic regularization and the Schroedinger bridge problem, will also be emphasized throughout the book.

We also completed a detailed outline for the book. There will be a core portion (Part 1), suitable for a one semester graduate course, covering the main, fundamental concepts which any researcher working on or with optimal transport will need to know, and a second part covering more specialized topics. After an overview, where static, dynamic and stochastic aspects of optimal transport are presented informally, the chapters in Part 1 will include (the plan is of course still tentative at this point):

1. Our introductory chapter on empirical measures.
2. Static optimal transport.
3. Wasserstein space.
4. Dynamic optimal transport.
5. Entropic regularization and stochastic optimal transport.
6. Further properties (including the geometry and PDE behind optimal transport).
7. Further geometry of Wasserstein space.
8. Gradient flows.

The chapters in Part 2 will be on:

9. Analysis and geometry of optimal transport.
10. Computational optimal transport.
11. A statistical perspective.
12. Multi-marginal optimal transport in physics, economics and operations research.
13. Constrained optimal transport and applications in finance and PDE.

There will also be appendices on important background topics, including measure theory and functional analysis, convex analysis, probability and statistics and partial differential equations; these will give readers convenient access to results from these areas which are required for the exposition in the rest of the textbook.

3 Outcome of the Meeting

The main anticipated outcome of this meeting will be the eventual completion and publication of our book. Two of us (Kim and Pal) are currently (in Fall, 2023) co-teaching a graduate level optimal transport class. They are using our outline as a template for the course and their notes will eventually be developed into part of the book. This also serves as a sort of dry run for the design vision, as topics and the way we plan to cover them can be adjusted if they do not work well in practice during the course. We plan to continue working on the details through the next year (2024) to have a draft of the book ready by the following year (2025).

References

[1] A. Galichon, *Optimam transport methods in economics*, Princeton University Press, Princeton, NJ, 2016.

[2] G. Peyre and M. Cuturi, *Computational Optimal Transport: With Applications to Data Science*, Foundations and Trends in Machine Learning, Now Publishers, 2019.

[3] F. Santambrogio, *Optimal Transport for Applied Mathematicians*, Progress in Nonlinear Differential Equations and Their Applications, Birkhauser Cham, 2015.

[4] C. Villani, *Topics in Optimal Transportation*, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2003.

[5] C. Villani, *Optimal transport: Old and new*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2009.