Approximation algorithms and the hardness of approximation

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1 Overview of the Field

Most of the discrete optimization problems arising in the sciences, engineering, and mathematics are NP-hard. This means that there exist no efficient algorithms to solve them optimally, assuming the $P \neq NP$ conjecture. The area of approximation algorithms focuses on the design and analysis of efficient algorithms that find solutions of cost within a guaranteed factor of the optimal cost. The area of hardness of approximation focuses on proving lower bounds on the guarantees that any efficient approximation algorithm can obtain for given problems assuming that $P \neq NP$ (or a similar complexity assumption). Over the last two decades, there have been major advances in the design and analysis of approximation algorithms, and in the complementary topic of hardness of approximation, see Vazirani [53], or Williamson and Shmoys [56].

The long-term agenda of our area is to classify all of the fundamental NP-hard problems according to their approximability and hardness thresholds. This agenda may seem far-fetched, but remarkable progress has been made over the last two decades. Approximation guarantees and matching hardness thresholds have been established for several key problems; e.g.

- covering and partitioning (the set covering problem, Feige [19]),
- algebra (overdetermined system of equations, Hastad [25])
- graphs (clique, colouring, Zuckerman [58]),
- optimization (maximum cut, Goemans and Williamson [22], Khot et al. [30]),

- constraint satisfaction (maximum SAT problems, Hastad [25]).

More significant than these specific successes is the impact of novel techniques on related areas of mathematics. We provide some examples.

1.1 Combinatorial Optimization:

The method of iterative rounding has been developed in the area of approximation algorithms to give remarkably good results for problems beyond the reach of classical combinatorial optimization, see Jain [27], and Lau et al. [35]. The technique has recently yielded elegant new proofs for a number classic results in combinatorial optimization.

1.2 Metric Embeddings:

Structure-preserving embeddings between various geometric spaces have been studied intensively for decades, in fields such as differential geometry and functional analysis. There are many applications of metric embeddings in the area of approximation algorithms. Moreover, the interaction between these fields has increased recently (see [37], [8], [7]).

1.3 Analysis of Boolean Functions:

Recent progress on hardness of approximation has come with the development of new tools in the area of "analysis of Boolean functions". This area combines techniques from harmonic analysis, probability theory, and functional analysis to study basic properties of Boolean functions. One recently developed tool, the Invariance Principle, has led to fruitful connections between hardness of approximation and the geometry of Gaussian space, see Mossel et al. [42].

2 Objectives of the Workshop Proposal

The goals of the workshop were as follows:

1. To bring together leading researchers in the fields of approximation algorithms and complexity theory, and to stimulate the exchange of ideas and techniques between the two groups.

2. To focus on a few key topics that could lead to deep new results in the areas of approximation algorithms, combinatorial optimization, hardness of approximation, and proof complexity. We describe a few topics below.

   (a) The most famous problem in all of discrete optimization is perhaps the Traveling Salesman Problem (TSP). Yet despite the attention paid to this problem, its approximability remains poorly understood. The best known approximation algorithm for the symmetric case is a classic $3/2$-approximation algorithm due to Christofides from 1976. On the other hand, the known hardness-of-approximation results are very weak.

   Over the last few years, there has been remarkable progress on several special cases of the TSP and on some closely related problems. Many of these advances are introducing new and very interesting connections between different areas such as probability, structural graph theory, coupled with technically difficult yet powerful new methods such as interlacing families of polynomials. In 2011 Oveis Gharan et al. [43] used properties of strongly Rayleigh measures together with an elaborate analysis of the structure of near-minimum cuts to obtain the first improvement on the $3/2$-approximation guarantee for a key special case of TSP called the graphic TSP. Since then, there has been a series of more work on this special case and related questions. The most recent result on this special case is a $7/5$-approximation algorithm of Sebo and Vygen [49] that hinges on a key probabilistic lemma of Momke and Svensson [41] coupled with an in-depth and novel analysis of structures that are well known in Combinatorial Optimization. An et al. [1] improved on a

20-year old $5/3$-approximation guarantee of Hoogeveen [26] for the s-t path TSP. Subsequently, Sebo [48] and Vygen [54] have improved on these results to obtain an 1.599-approximation guarantee, by using further probabilistic insights. More recently, Gottschalk and Vygen [23] and Sebo and van Zuylen obtained 1.566 and 1.529 approximations for the s,t-path TSP problem, respectively. Relying on (and extending) the major result by Marcus, Spielman, and Srivastava [39] that proves a conjecture of Kadison-Singer, Anari and Oveis Gharan [2] recently showed the existence of $O(\mathrm{poly}(\log\log)n)$-thin spanning trees. The result implies an $O(\mathrm{poly}(\log\log)n)$ upper bound on the integrality gap of the Held-Karp LP relaxation for the asymmetric TSP (improving the $O(\log n/\log\log n)$ bound from 2010 [9]). The LP is long conjectured to have an $O(1)$ integrality gap. The result of [2], however, does not imply an approximation algorithm, it only provides an estimate of the optimum value. Almost concurrently, Svensson [51] showed that for the case of shortest path metrics of directed graphs (graphic ATSP), the integrality gap of the Held-Karp LP is $O(1)$ and provides an efficient algorithm for it. The two biggest open problems in this area remain to improve upon the $3/2$-approximation for TSP and to obtain a constant factor approximation for ATSP. By re-focusing attention on this problem, our goal is to continue the momentum from the past two workshops. Two notable new results in this area were found very recently, after our BIRS workshop: Vygen and Traub [55] recently presented a $1.5+\epsilon$ approximation for s,t-path TSP, nearly matching the performance ratio of Christofides' algorithm for metric TSP. Svensson, Tarnawski, and Vegh [52] presented a constant factor approximation for the asymmetric TSP problem; their bounds are relative to the standard ATSP LP relaxation, confirming the conjecture that it has constant integrality gap.

(b) The Unique Games Conjecture (UGC) which was posed in 2002 by Khot [29] and the implications of it have attracted a lot of attention over the last 13 years. The conjecture states that a certain type of constraint satisfaction problem is hard to approximate. If the conjecture is true, it shows that many of the approximation algorithms we have (in particular SDP based algorithms) are best possible ([45, 31, 30]). More specifically UGC implies near tight approximatibility thresholds for a large class of constraint satisfaction problems (CSPs) among others (see Raghavendra [46]). In a sense, UGC predicts that there is a "meta-algorithm" that is optimal for those problems and this meta-algorithm is based on SDP [10]. Refuting the conjecture would most likely require designing new algorithmic techniques that could potentially lead to improved approximation algorithms for many other problems. One component of the workshop will focus on this conjecture and surrounding issues in the complexity of optimization problems.

Lasserre hierarchy / Sum-of-Squares algorithms:

Use of semidefinite programming (SDP) relaxations and the lift-and-project strengthening of them has attracted a lot of attention in the field in the last decade or so. The Lasserre hierarchy is a systematic method of strengthening SDP relaxations by adding more constraints. In some instances these methods have been successfully applied to obtain improved approximation algorithms for some classical results (e.g. [15]). More importantly, although some results support the UGC, some recent works have cast more doubts on it using Lasserre SDP based algorithms. For example, Arora et al. [6] shown that the powerful Lasserre SDP hierarchy of algorithms could be used to obtain a subexponential-time algorithm for Unique Games (UG). More recently, Barak et al. [11] have used the connection between Lasserre algorithms and Sum-of-Squares (SOS) proof complexity, and have shown that the known hard instances of the UG problem can be analyzed by constant-degree SOS proofs, and thus be solved efficiently.

Extended formulations and their complexity:

Feasible solutions to instances of combinatorial optimization problems often naturally correspond to the vertices of certain polyhedra. One way of designing an efficient algorithm for a given optimization problem is therefore to find a compact description for the associated polyhedron, and to then apply an efficient LP algorithm. In ground-breaking work Yannakakis [57] first showed that for the TSP, every symmetric LP formulation must have an exponential number of constraints. Symmetry here means that for every permutation of cities there is a corresponding permutation of the variables that leaves the LP invariant. Fiorini et al. [21] recently resolved Yannakakis' main open problem and showed that TSP has no symmetric or asymmetric polynomial-sized formulation. In another breakthrough, Rothvoss [47] recently showed that no subexponential-size extended formulation can exist for the matching polyhedron either.

(c) Routing problems in graphs arise in many areas of computers science, from VLSI design to Robotics. They have also been extensively studied in the graph theory community. Two of the most basic graph routing problems are the Edge Disjoint Paths (EDP) problem and Congestion Minimization. In EDP, we have to route a maximum number of demand pairs from a given collection in a graph via disjoint paths. In Congestion Minimization, all demand pairs must be routed, while minimizing the maximum load on any edge. Both problems are still poorly understood: for EDP the best upper and lower bounds have ratios $O(n^{1/2})$ [32] and $\Omega(\log^{1/2} n)$ [4, 3], while the upper and lower bounds for congestion minimization have ratios $O(\log n/\log\log n)$ [44] and (roughly) $\Omega(\log\log n)$ [5], respectively. If one allows up to 2 paths to share an edge, a polylogarithmic approximation was recently shown [16, 17]. Graph routing problems are naturally closely related to network cuts and flows. The new techniques for graph decomposition introduced in [16] have lead to new results in several other areas, such as a polynomial bound for the Excluded Grid Theorem of Robertson and Seymour [14]. Another closely related topic is graph sparsification: given a graph G and a small subset T of its vertices, called terminals, we would like to "compress" G into a much smaller graph H that contains the vertices of T, so that H behaves similarly to G with respect to the terminals. Graph sparsifiers naturally arise in approximation algorithms, graph theory, and fixed parameter tractability, and they have been studied in all these communities, often independently. If we require that the sparsifier H only contains the terminals, then there are known constructions that achieve quality (approximation factor) $O(\log k/\log\log k)$ for both the cut and the flow sparsifiers [40, 36, 13, 38, 18], and it is known that no better than $\Omega(\sqrt{\log n})$-quality is achievable for this setting [20, 38, 13, 18]. If H is allowed to contain additional vertices, better results (namely constant-quality) are known. Unfortunately, we still do not know whether it is possible to construct constant-quality cut and flow sparsifiers whose size only depends on k. Some of these recent results rely on graph decomposition techniques that were developed in the area of approximation algorithms for graph routing problems. Quality-1 cut sparsifiers were introduced under the name of mimicking networks by Hagerup et al. [24], and they have been used to provide kernels for various cut problems, such as, for example, minimum multiway cut [33]. However, the best current upper and lower bounds on the size of a mimicking network are $2^{2^{O(k)}}$ [24, 28, 12] and $2^{\Omega(k)}$ [34, 28], respectively, which show that even this very basic problem is still not well understood.

3 Presentation Highlights

3.1 Bimodular Integer Linear Programming

The first plenary talk was by Rico Zenklusen. He showed how one can solve any integer linear program (ILP) defined by a constraint matrix whose sub-determinants are all within $\{-2,-1,0,1,2\}$ in strongly polynomial time. This is a very nice extension of the well-known fact that ILP's with totally unimodular (TU) constraint matrix are solvable in strongly polynomial time. This result uses several techniques. They first reduce the problem to a particular parity-constrained ILP over a TU constraint matrix and then use Seymour's decomposition of TU matrices to break this ILP into simpler base problems. Then they show how these simpler problems can be solved using combinatorial optimization techniques. He also highlighted some open problems in this field and some possible extensions to larger classes of ILPs.

3.2 A Simply Exponential Upper Bound on the Maximum Number of Stable Matchings

In the 2nd plenary talk, Shayan Oveis Gharan highlighted their recent result on the number of stable matchings, which is a classical problem. They show that for any stable matching instance with n men and n women the number of stable matchings is at most C^n for some universal constant $C > 1$. The proof is based on a reduction to counting the number of down-sets of a family of posets that we call mixing.

3.3 Approximating spanners and distance oracles

Despite significant recent progress on approximating graph spanners (subgraphs which approximately- preserve distances), there are still several large gaps in our understanding. In the 3rd plenary talk, Michael Dinitz did a survey of some recent results, with a focus on low-stretch spanners (from SODA '16) and on spanners with demands (from SODA '17). He described the gaps and open problems which remain, including open questions on the power of linear and semidefinite relaxations for these kinds of network design problems. He also presented some recent results on approximation algorithms for optimizing distance oracles (the natural data-structure version of spanners). The talk mostly focused on the many interesting open problems remaining in approximation algorithms for optimizing data structures.

3.4 Approximation Schemes for Clustering Problems: Now With Outliers

Recent developments in local search analysis have yielded the first polynomial-time approximation schemes for the k-Means, k-Median, and Uncapacitated Facility Location problems (among others) in a variety of specific classes of metrics . An important extension of these problems is to the setting with outliers. That is, we we are given an additional parameter Z and may discard up to Z points/clients in the input. This is especially important in the setting of k-Means clustering where even a small fraction of outliers may cause a noticeable deviation in the centroids of near-optimum solutions. In the 4th plenary talk, Zac Friggstad started with a brief review of their recent work from last year in local search analysis for k-Means clustering in Euclidean metrics. Then he presented a more recent development: a general framework for adapting local search analysis for clustering problems to get approximations for their variants with outliers. In particular, how they obtain the following results for clustering in doubling metrics (including constant-dimensional Euclidean metrics) and shortest path metrics of bounded genus graphs. 1) PTASes for uniform opening cost UFL with outliers and 2) bicriteria PTASes that open $(1+\epsilon)k$ centres for k-Median and k-Means clustering with outliers. There is no violation on the given bound for outliers in any of these approximations.

3.5 The Paulsen problem, continuous operator scaling, and smoothed analysis

The Paulsen problem is a basic open problem in operator theory: Given vectors $u_1, ..., u_n$ in R^d that are eps-close to satisfying the Parseval's condition and the equal norm condition, is it close to a set of vectors $v_1, ..., v_n$ in R^d that exactly satisfy the Parseval's condition and the equal norm condition. Given $u_1, ..., u_n$, the squared distance (to the set of exact solutions) is defined as $\inf_v \sum_{i=1}^n normu_i - v_{i2}^2$ where the infimum is over the set of exact solutions. Previous results show that the squared distance of any eps-close solution is at most $O(poly(d, n, \epsilon))$ and there are eps-close solutions with squared distance at least $Omega(d\epsilon)$. The fundamental open question is whether the squared distance can be independent of the number of vectors n.

In the last plenary talk, Lap Chi Lau showed how they answer this question affirmatively by proving that the squared distance of any eps-close solution is $O(d^7\epsilon)$. Their approach is based on a continuous version of the operator scaling algorithm and consists of two parts. First, they define a dynamical system based on operator scaling and use it to prove that the squared distance of any eps-close solution is $O(d^2n\epsilon)$. Then, they show that by randomly perturbing the input vectors, the dynamical system will converge faster and the squared distance of an eps-close solution is $O(d^3\epsilon)$ when n is large enough and eps is small enough. To analyze the convergence of the dynamical system, they develop some new techniques in lower bounding the operator capacity, a concept introduced by Gurvits to analyzing the operator scaling algorithm.

4 Scientific progress made and outcome of the meeting

The schedule of the workshop provided ample free time for participants to work on joint research projects. A number of new research projects were initiated during the workshop, while some other researchers used the opportunity to continue to work on projects started earlier. The research talks and the plenary talks were very well received.

Rico Zenklusen reports that during the workshop he and his colleagues (Chaitanya Swamy, Andr Linhares, and Neil Olver) made progress on a project on optimizing over the intersection of matroids. In particular, at BIRS he and Swamy spent time during the workshop on this project and they found an interesting extension and application of a technique we developed earlier. In particular, further discussions with Lap-Chi that he had at BIRS proved very helpful to find this connection. This revived a project on which they worked for some time already, but had trouble to identify a good way to fully exploit their techniques.

Also, Mohammad Salavatipour and his Ph.D. student Mirmahdi Rahgoshay (who participated at the workshop) started a project with Rico Zenklusen on a problem related to resource management on a network (with application to fire containment and spread of other harmful events). So far they have been able to improve the previously best known results (a 12-approximation by Zenklusen in SODA17) to an asymptotic approximation scheme that runs in quasi-polynomial time. The project is on-going with the goal of obtaining a true polynomial time approximation scheme.

Sam Hopkins (another participant) reports that he made significant progress on two projects as a result of the workshop. At the workshop he gave the first public talk on his recent work with Jerry Li on clustering via sum of squares proofs; discussions with colleagues afterwards and questions during the talk helped clarify a number of points in the new paper and its relationship to previous work. In particular Aravindan Vijayaraghavan and Sam had some substantial discussions on these matters. He also had the opportunity to continue an existing project with Tselil Schramm on integrality gaps for linear programming hierarchies, on which we made substantial progress.

Andreas Wiese reports that he and Fabrizio Grandoni continued their collaboration on a problem related to the Unsplittable Flow on a Path problem. Their discussions were very fruitful, in particular they were able to find the best possible approximation factor for an important special case. Moreover, they could simplify some of their argumentations which will yield a cleaner presentation in the paper they plan to write on the topic.

Fabrizio Grandoni reports: "I continued my collaboration with Parinya Chalermsook and Bundit Laekhanukit about the Group Steiner Tree and related problems. The discussions that we had during the workshop were very fruitful and might eventually lead to some concrete progress on the problems that we are studying."

Tselil Schramm tells us: "During the workshop, Sam Hopkins and myself continued working on an ongoing project, trying to use the recent "pseudocalibration" technique to improve Sherali-Adams lower bounds for max-cut and other constraint satisfaction problems. I was also exposed to the Paulsen problem by Lap Chi Lau's excellent talk, and later began working on the problem (trying to improve on Lap Chi and coauthors' result)."

Also, Viswanath Nagarajan reports: "I continued my collaboration with Anupam Gupta on designing approximation algorithms for stochastic load balancing. The breakout times during the workshop were very useful in continuing our research discussions on this topic. We were able to come up with a counter-example for one of approaches. This has been useful for us in identifying an alternative approach, which is still work in progress. "

The above are only a few examples of the research progress made during or after the workshop, and there are other ongoing projects that started at the workshop.

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