Lattice walks at the Interface of Algebra, Analysis and Combinatorics

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1 Overview of the field

1.1 Lattice Path Models

Lattice paths are a classic object of mathematics, with applications in a wide range of areas including combinatorics, theoretical computer science and queuing theory. In the past ten years, several new approaches have emerged to determine formulas for exact enumeration. In particular, several systematic methods have arisen from very disparate areas of mathematics. The study of lattice walks restricted to the first quadrant is surprisingly illuminating; it reveals connections between abstract combinatorial structures and classes of analytic functions.

Here we define a *lattice path model* to be a combinatorial class of walks on the integer lattice, typically starting at the origin, and remaining in some cone. It is prescribed by a finite set of vectors $\mathcal{S}$, the allowable steps one can take in the lattice. Much attention has been paid to walks with small steps, where $\mathcal{S} \subseteq \{-1, 0, 1\}^2$, constrained to the quarter plane, $\mathbb{Z}^2_{\geq 0} = \{(x, y) \in \mathbb{Z}^2 : x \geq 0, y \geq 0\}$. We frequently abbreviate the small steps using compass directions or vector diagrams. A *walk* of length n which starts at (x_0, y_0) is a sequence of steps $(s_1, \dots s_n)$ such that $s_i \in \mathcal{S}$ and all points $\{(x_0, y_0) + \sum_{i=1}^{\ell} s_i : 0 \leq \ell \leq n\}$ remain in the designated cone. Figure 1 is a walk from the quarter plane model with step set $\mathcal{S} = \{(0, 1), (0, -1), (1, 0), (-1, 0)\}$.

The basic enumerative problem is to determine formulas (exact or asymptotic) for the number of walks of length n in a given model. Early studies [24, 21] determined formulas for particular models, generally using adhoc methods. Fayolle, Iagnogorski and Malyshev completed a comprehensive study of stationary distribution of small step models in the quarter plane. They built a functional equation, and provided an analytic toolkit to resolve cases. The work inspired Bousquet-Mélou, who adapted their methods for exact enumeration. She was able to directly derive the results of Kreweras and Gessel in a straightforward manner. This approach was adapted to many models by Bousquet-Mélou and Mishna [12], who provided a structured analysis of the generating functions for the small step models restricted to the first quadrant.

Specifically, they determined that of the $2^8 - 1$ possible small step models, there were precisely 79 that were combinatorially distinct and nontrivial. They provided generating function expressions for 22 of them, and made conjectures on the nature of the generating functions of the rest. The conjectures use a related group defined for each step set (inspired by the work of Fayolle et al.) This presented a tidy, and clearly intriguing, classification challenge.

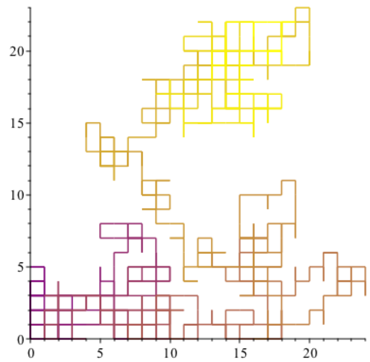

Figure 1: A walk of length 1000 from the simple quarter plane model with step set $\mathcal{S} = \{(-1,0),(1,0),(0,-1),(0,1)\}$

A wide variety of techniques have been developed in the interim, and this meeting sought to bring researchers together to discuss commonalities, feasible extensions, and future directions. The approaches discussed at this meeting included bijective methods, probability, computer algebra, complex analysis, algebraic geometry, and differential Galois theory. There are many references for these methods [12, 4, 19, 7, 27, 16]. In this document we outline some of these topics as they apply to lattice path enumeration, and discuss the implications in generating function classification.

1.2 The generating function and its kernel

We revisit the idea of more general cones towards the end of the document. For now, consider the quarter plane and a fixed given step set $\mathcal{S}$. Let $e_{(k,\ell)\to(k',\ell')}(n)$ denote the number of walks in the cone starting at (k,ℓ), ending at (k',ℓ'), taking n steps, each from $\mathcal{S}$. The *complete generating function* for these numbers is the following series, viewed to be an element of $\mathbb{Q}[x,y][\![t]\!]$:

$$\sum_{k,\ell,n\geq 0} e_{(0,0)\to(k,\ell)}(n)\, x^k\, y^\ell t^n.$$

Formally, weights on the steps add no significant difficulty to the set up. Each step is assigned a (real, positive) weight, and the weight of a walk is the product of weights of its component steps. Remark, we recover the unweighted model by setting the weight of each step to be 1 (and hence the weight of each walk is 1). Alternatively, if the sum of the weights is 1, we describe a probability model. More precisely, let $(d_{i,j})_{(i,j)\in\{0,\pm1\}^2}$ be a family of elements of $\mathbb{Q}\cap[0,1]$ such that $\sum_{i,j} d_{i,j} = 1$. The weight $d_{i,j}$ can be viewed as the probability that a walk take a step in the direction (i,j), with $d_{0,0}$ the probability that it does not move.

For any $(i,j)\in\mathbb{Z}^2_{\geq 0}$ and any $n\in\mathbb{Z}_{\geq 0}$, we let $q_{i,j}(n)$ be the sum of the weights of all walks of length n starting at the origin, and ending at the point (i,j). Our main focus here is the trivariate generating series

$$Q(x,y;t) := \sum_{i,j} q_{i,j}(n) x^i y^j t^n. \tag{1}$$

It is sufficient to consider the case $\sum_{i,j} d_{i,j} = 1$, by suitably scaling the t variable (although many techniques are custom made for the unweighted version). Then, for any $n\in\mathbb{Z}_{\geq 0}$, $|q_{i,j}(n)| \leq \sum_{i,j} |q_{i,j}(n)| \leq (\sum_{i,j} |d_{i,j}|)^n = 1$. We deduce that $Q(x,y;t)$ converges for all $(x,y;t)\in\mathbb{C}^3$ such that $|x|<1$, $|y|<1$ and $|t|\leq 1$.

The *inventory* of the step set is a Laurent polynomial which we might write as a function of either variable x or y:

$$\begin{aligned} S(x,y) &= \textstyle\sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j} x^i y^j \\ &= A_{-1}(x)\frac{1}{y} + A_0(x) + A_1(x) y \\ &= B_{-1}(y)\frac{1}{x} + B_0(y) + B_1(y) x. \end{aligned} \tag{2}$$

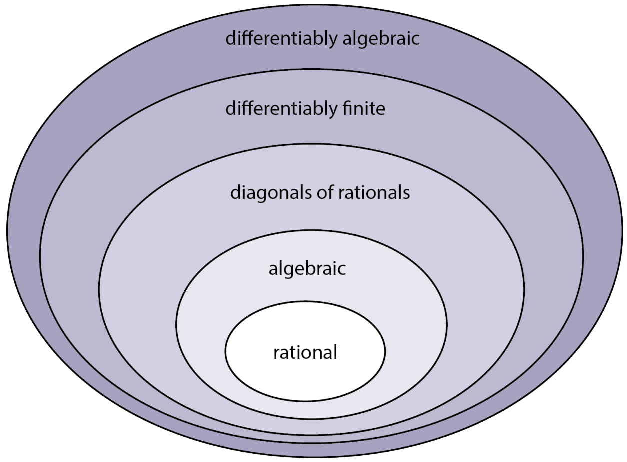

Figure 2: Different classes of generating functions

Here $A_i(x) \in x^{-1}\mathbb{Q}[x]$, and $B_i(y) \in y^{-1}\mathbb{Q}[y]$. The *kernel* of the walk is defined by

$$K(x, y, t) := xy(1 - tS(x, y)). \tag{3}$$

A walk is either a walk of length 0, or a walk followed by a valid step. This straightforward decomposition has a natural translation into generating functions, into what we call the *kernel equation*. A property specific to models of walks in the quarter plane is that the generating function for walks that end on a boundary can be obtained from an evaluation of $Q(x, y; t)$. For example, the family of walks that begin and end at the origin have generating function $\sum_{n\geq 0} q_{0,0}(n)t^n = Q(0, 0; t)$. The walks that end on the y-axis have generating function $\sum_{n,k\geq 0} q_{k,0}(n)x^k t^n = Q(x, 0; t)$. This is a particularly convenient observation.

Following [18, Section 1], and translating the aforementioned combinatorial recurrence, we may prove that the generating series $Q(x, y, t)$ satisfies the following functional equation:

$$K(x, y, t)Q(x, y; t) = xy - R(x, t) - S(y, t) + td_{-1,-1}Q(0, 0, t) \tag{4}$$

where

$$R(x, t) := K(x, 0, t)Q(x, 0; t), \quad \text{and} \quad S(y, t) := K(0, y, t)Q(0, y; t). \tag{5}$$

Many enumeration strategies start with Equation (4), and then differ in their subsequent manipulations. This report summarizes only some of them. Central to many of these methods is the property of small step walks is that $K(x, y; t)$ is quadratic. Considering larger steps is considerably more complicated. Nonetheless, stepsets with negative steps of length 2 have recently been thoroughly considered [10].

1.3 Classification of generating functions

Lattice walks provide a unified context to examine combinatorial origins of different function classes. The combinatorial complexity of a given model is correlated to the analytic complexity of its generating function. It remains open to completely formalize many of the known connections.

Rational A generating function is rational if it is the quotient of two polynomials. Generating functions of regular languages are rational. The coefficients of rational series, and their asymptotics are extremely well understood, see [20] for a pedagogical treatment. For example, the the quarter plane model with step set $\{N, NE, E\}$ is unconstrained, and consequently it is easy to show the generating function is rational: $Q_{N,NE,E}(x, y; t) = \frac{1}{1-t(x+xy+y)}$.

Algebraic An univariate generating function $f(t)$ is algebraic if there is a polynomial $P(u, v) \in \mathbb{C}[u, v]$ such that $P(f(t), t) = 0$. The polynomial P provides a finite encoding for $f(t)$, and can be used in computations and closure properties. The asymptotics are also relatively well understood. A multivariate function of m varables is algebraic if it solves a polynomial equation in $m + 1$ variables. Rational functions are algebraic.

From the combinatorial perspective, combinatorial classes such as trees and maps possess the archetypal algebraic structure, and have algebraic generating functions.

In the realm of lattice models, unidimensional walks have algebraic generating functions, and many two-dimensional models can be reduced to this case. Expressions for the generating functions, and the coefficient asymptotics are straightforward to compute.

Transcendental D-finite A univariate function is differentiably finite, commonly called *D-finite* or *holonomic*, if it satisfies a linear differential equation with polynomial coefficients. In 1980 Stanley highlighted the interest to combinatorialists by pointing out the very combinatorial collection of closure properties: including differentiation, algebraic substitution and Hadamard product. It remains open to find a combinatorial mechanism that adequately captures this class of functions. It strictly contains algebraic functions.

The following result (compiled from work of Kath, André, and Garoufalidis) is useful to show that a function is not D-finite.

Theorem 1. *If a series $\sum a_n t^n$ in $\mathbb{Z}[t]$ is D-finite, with radius of convergence in $(0, \infty)$, then its singular points are regular with rational exponents. Consequently, the asymptotic expansion has the form of a finite sum*

$$a_n \sim \sum \lambda^{-n} n^{\alpha} \log^k(n) f_{\lambda,\alpha,k}(1/n)$$

where α is rational, λ is algebraic, k is a nonnegative integer, and f is a polynomial over $\mathbb{Q}[x]$.

For example, Bostan, Raschel and Salvy proved that several models of excursions have non-D-finite generating functions because the sub-exponential growth is irrational [7].

A multivariate function is D-finite with respect to a set of variables if it is D-finite with respect to each of the variables.

Diagonals The *diagonal* of a formal power series $\sum_{i_1,\ldots,i_d \geq 0} f_{i_1,\ldots,i_d} z_1^{i_1} \cdots z_d^{i_d}$ is the series $(\Delta F)(t) = \sum_{n\geq 0} f_{n,\ldots,n} t^n$.

The diagonal of a multivariate function is defined for a particular series expansion of that function around the origin.

Furstenburg showed in 1967 that algebraic functions can be expressed as a diagonal of a bivariate rational function by providing the rational based on the polynomial that the function satisfies. Lipshitz showed that the diagonal of a D-finite multivariate series was D-finite. There do exist D-finite functions which could not be a diagonal of a rational, for example e^x, but perhaps if the conditions are right, there might be an equivalence. The following conjecture has been considered for a while:

Open problem 1 (Christol, 1990)**.** Can any D-finite series with integer coefficients and a positive radius of convergence can be expressed as a diagonal of a rational series?

Differentiably algebraic A function is *differentiably algebraic*, or D-algebraic, if there exists an algebraic relationship between $f(x)$ and its derivatives. More precisely, there exists a polynomial $P \in \mathbb{C}[x]$ so that

$$P(f(x), f'(x), \ldots, f^{(k)}(x)) = 0.$$

This class clearly contains the D-finite functions, but is otherwise less well understood.

Beyond this Finally, we say that the remaining functions are *hyper transcendental*.

Class	Count	Step Sets
Algebraic	4	
Transcendental D-finite	19	
Non-D-finite D-algebraic	9	
Hyper transcendental	42	
D-transcendental	5	

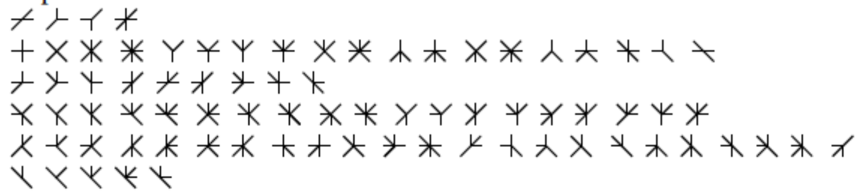

Figure 3: The nature of the complete generating function $Q(x, y; t)$ for the 79 small step quarter plane lattice path models

1.4 Classifying Lattice paths

So where do the generating functions of the 79 small step quarter plane models fall? Attempts to answer this question have spawned a very active area of research. The answers provide insight on the subtle differences of the function classes. Figure 3 is a compilation of results from numerous articles.

Slightly less is known about the univariate counting generating functions. Since these classes are closed under algebraic substitutions, which includes evaluations, we can deduce some containments, but we cannot conclude that many of the step sets with non-D-finite complete generating function have a non-D-finite counting generating function.

Open problem 2. Describe combinatorial criteria on 2D lattice models which determine the nature of the generating functions. Which criteria hold in problems of higher dimension?

Open problem 3. Prove the remaining unclassified small step univariate counting generating functions non-D-finite.

2 Computer Algebra

Part of the appeal of lattice path models is the ease with which one can experiment using computer algebra tools. It is relatively straightfoward to develop a series expansion from the basic recurrence, and then form hypotheses and conjectures about the generating functions.

2.1 Guess ’n’ prove

Software to guess the algebraic or differential equation satisfied by a series has been used enumerative combinatorics for nearly 30 years. The main implementions work by generating a high order truncation of the (possibly multivariate) generating function, followed by a guessing stage which tries to fit this truncation into algebraic or differential equations of various orders and degrees using Padé approximants. In Maple, this is implemented in the gfun [30] package.

The equations that are found remain conjectural, until they are proved. In this field, the most famous example, perhaps, is known as “Gessel’s walk”. In 2000 Ira Gessel, via personal communication, suggested that the generating function for the quarter plane walks with step set $\{SW, W, NE, E\}$ should be D-finite. This was based on the observation that the counting sequence for excursions, that is, the walks that start and end at the origin, appeared to fit very nice formula. This lead to a very vigourous discussion, as the other models with D-finite generating functions were easily guessed to be, and this model resisted the approach.

The D-finiteness of the excursion generating function was published in 2009 [23]. The proof used a cutting edge holonomic systems approach, and creative telescoping to confirm Gessel’s conjecture. Still, it remained open to determine the nature of the complete generating function.

Around the same time, Bostan and Kauers [4] used the aforementioned approach based on Padé-Hermite approximants to systematically guess algebraic and differential equations satisfied by the 79 non-isomorphic two dimensional models in the quarter plane. In addition to detailing several algebraic and arithmetic conditions which help one believe that a guessed differential equation truely annihilates the multivariate generating function, the authors produced an influential table of guessed asymptotics for the 23 models they predicted to have D-finite generating function.

It surprised many when Bostan and Kauers [5] proved not only the D-finiteness, but the *algebraicity* of the multivariate Gessel generating function $Q(x, y; t)$! Their approach rigorously certified a guessed minimal polynomial of $Q(x, y; t)$ using algebraic elimination theory. The result was a computational tour de force – the algebraic equation that $Q(x, y; t)$ satisfies requires several MB to store. Researchers were challenged to demonstrate the algebraicity using "human" techniques. The community delivered, and such results appeared in the following few years. Indeed, resolving this problem is a recurrent theme. For more details, see the approachable survey of Bostan and Raschel [6].

Computer algebra methods have proved indispensible for providing bulk results. Notably, Bostan, Raschel and Salvy [7] iterated through to show that the excursions in 56 of the models did not have a D-finite generating function because the asymptotic expressions were incompatible. Specifically, the subexponential growth had an irrational exponent.

Recently, Bostan et al. [11] proved all guessed annihilating algebraic and differential equations for the 23 D-finite models using creative telescoping on diagonal expressions, and wrote lattice path generating functions as explicit integrals of hypergeometric functions. These expressions, in turn, allowed the authors to classify all transcendental D-finite generating functions, and determine which combinatorially interesting specializations of the transcendental generating functions are algebraic.

In this workshop, the topic was introduced by a thorough mini-course given by Manuel Kauers. Several talks addressed computational aspects, including how to prove the transcendance of a D-finite generating function from an annihilating differential equation.

This is but a small snapshot of the computer algebra methods in this field. A far more complete and detailed summary is the habilitation thesis of Bostan [8].

Open problem 4. The computational methods of Bostan et al. [9] show that the number of excursions on the step set $\{(-1, 0), (-1, 0), (-1, 1), (-1, -1), (1, 0), (1, 1)\}$ (note the double $(-1, 0)$ step) satisfy

$$a_{2n} = \frac{6(6n+1)!(2n+1)!}{(3n)!(4n+3)!(n+1)!}$$

(there are no excursions of odd length). Find a 'human' proof of this result.

Open problem 5. The four quadrant models with algebraic multivariate generating functions have zero orbit-sum, meaning the kernel method does not allow one to represent these generating functions as diagonals of rational functions. As algebraic functions, they can be written as diagonals of bivariate rational functions through known constructions, but these representations are unweildy and hard to analyze compared to the nice 'combinatorial' expressions arising from the kernel method. Do there exist simple 'combinatorial' rational functions whose diagonals give the generating functions of the 4 models with algebraic generating function?

2.2 Enhanced data manipulation

The strategies for guessing apprimations of asymptotic formulas from initial series data are becoming increasingly sophisticated. If one assumes that the generating function $f(z) = \sum a_n z^n$ has a power law singularity of the form

$$f(x) \sim C\left(1 - \frac{x}{x_0}\right)^{\alpha},$$

the series can be analyzed to produce high precision numerical estimates for the unknown quantities. In this workshop, Tony Guttman talked about how to use differential approximants to predict subsequent terms from some initial data. Every differential approximant naturally reproduces exactly all coefficients used in its derivation. Being a D-finite differential equation, it implies the value of all subsequent coefficients. These subsequent coefficients will usually be approximate.

These techniques have been used by the Melbourne school for a variety of combinatorial problems, and recent work by Guttman and Elvey Price on the enumeration of Planar Eulerian Orientations [17] was presented at this workshop.

3 The Kernel Method

Kernel equations here denote functional equations of a particular form that arise from combinatorial recursions. One separate the terms into two classes: on the left hand side is an expression with the complete generating function, known in some circles as the *bulk*. One the right hand side, there are functions which involve the boundary or more generally, evaluations of the complete generating function. The kernel determines the bulk behaviour. The two main ways to proceed are to manipulate the equation to either eliminate the boundary conditions, or to eliminate the bulk.

Andrew Rechnitzer provided a short course on variants of the *kernel method* for lattice walks, with a view to the study of polymers. He identified the main steps:

- identify a catalytic or auxillary variable to track a useful parameter;
- use a combinatorial recursion to establish a functional equation for the complete generating function;
- rearrange the terms and write the equation in kernel form;
- evaluate the auxiliary variables to either eliminate the kernel, or to eliminate the boundary terms;
- extract the complete generating function.

Remark the two principle variants: the first one uses substitutes solutions to the kernel into the main equation, and manipulates the boundary terms ("eliminate the bulk"); the second uses invariants of the kernel to eliminate the boundary terms ("eliminate the boundary").

Eliminating the bulk can find explicit generating expressions. In polymer modelling problems, it is used to understand the behaviour for different evaluations of the auxiliary parameter. A typical example is contacts with a boundary, well explained by the talk of Aleks Owczarek on three interacting friendly walks [31]. This is also central to several analytic approaches which we describe below.

Eliminating the boundary using invariants of the kernel has been very useful for lattice path enumeration. The finiteness of a subgroup of the kernel invariants is implicated in the classification of the complete generating function. Bousquet-Mélou and Mishna described how to use such invariants to express the complete generating function as a sub-series of a Laurent series expansion of a computable rational function.

4 Analytic combinatorics of several variables

Complex analysis methods to determine asymptoic enumeration formulas for combinatorial classes have become standard [20]. Initially the majority of the result came from the singularity analysis of of univariate generating functions. The multivariate problem is substantially harder, but steady progress over the past two decades has lead to some systematic strategies. Relevant here has been the work developed by Pemantle, Wilson, Baryshnikov, and coauthors under the name of Analytic Combinatorics in Several Variables, abreviated (ACSV) [28].

Melczer and Mishna [26] noted the applicability of these methods to lattice enumeration. They restricted to models with significant symmetry, but were able to generalize the strategy to higher dimensions. Melczer and Wilson performed a case by case analysis [27] to determine asymptotic formulas for most of the small step 2D models with a finite group. Courtiel et al. [15] used ACSV to determine asymptotic formulas for the weighted Gouyou-Beauchamps model.

5 Differential Galois Theory

The strategy to eliminate the "bulk" leads to important theoretical results. The key is to understand that the kernel is eliminated on an elliptic curve. Charlotte Hardouin gave an excellent course on how to apply methods in differential Galois theory to prove the hypertranscendance of lattice model generating functions. Similar strategies have been used to show the transcendance of the Gamma function.

We first fix the value of t, admittedly a nontraditional strategy for generating function combinatorics. Specifically, let us fix $0 < t < 1$ with $t \notin \overline{\mathbb{Q}}$.

We start from Equation (4)

$$K(x,y,t)Q(x,y;t) = xy - R(x,t) - S(y,t) + td_{-1,-1}Q(0,0,t). \tag{6}$$

Consider the algebraic curve $\overline{E_t}$, which is defined as the zero set in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ of the following homogeneous polynomial

$$\overline{K}(x_0,x_1,y_0,y_1,t) = x_0x_1y_0y_1 - t\sum_{i,j=0}^{2} d_{i-1,j-1}x_0^i x_1^{2-i} y_0^j y_1^{2-j} = x_1^2y_1^2 K(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t). \tag{7}$$

This curve is an *an elliptic curve*. This is the situation studied in [12, 25], and for the Galoisian point of view, see [16].

Thanks to uniformization, we can identify E_t with $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Specifically, consider the map

$$\begin{array}{ccc} \mathbb{C} & \rightarrow & \overline{E}_t \\ \omega & \mapsto & (\mathfrak{q}_1(\omega), \mathfrak{q}_2(\omega)), \end{array}$$

where $\mathfrak{q}_1, \mathfrak{q}_2$ are rational functions of $\mathfrak{p}$ and its derivative $d\mathfrak{p}/d\omega$, $\mathfrak{p}$ the Weirestrass function associated with the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (cf. [25, Section 3.2]). Therefore one can lift the functions $R(x,t)$ and $S(y,t)$ to functions $r_x(\omega) = R(\mathfrak{q}_1(\omega), t)$ and $r_y(\omega) = S(\mathfrak{q}_2(\omega), t)$.

One can deduce from [25, Theorems 3 and 4], that the functions $r_x(\omega)$ and $r_y(\omega)$ can be continued meromorphically as univalent functions on the universal cover $\mathbb{C}$. Furthermore, for any $\omega \in \mathbb{C}$, we have

$$\begin{aligned} \tau(r_x(\omega)) - r_x(\omega) &= b_1, \text{ where } b_1 = \iota_1(\mathfrak{q}_2(\omega))(\tau(\mathfrak{q}_1(\omega)) - \mathfrak{q}_1(\omega)) && (8)\\ \tau(r_y(\omega)) - r_y(\omega) &= b_2, \text{ where } b_2 = \mathfrak{q}_1(\omega)(\iota_1(\mathfrak{q}_2(\omega)) - \mathfrak{q}_2(\omega)) && (9)\\ r_x(\omega + \omega_1) &= r_x(\omega) && (10)\\ r_y(\omega + \omega_1) &= r_y(\omega). && (11) \end{aligned}$$

where τ is the automorphism of the field of meromorphic functions sending $f(\omega)$ onto $f(\omega+\omega_3)$ and ω_3 is explicitely given in [25, Section 3.2].

One can show that $r_x(\omega)$ is differentially algebraic with respect to $\frac{d}{d\omega}$ over C_E, the field of elliptic functions with respect to $\overline{E_t}$ if and only if $Q(x,0,t)$ is differentially algebraic with respect to $\frac{d}{dx}$ over $\mathbb{C}(x)$.

5.1 The order of the differential operator

In [16], it is shown that if $r_x(\omega)$ is differentially algebraic, then there is a second order linear differential operator L with constant coefficients such that $L(r_x) - g$ is an ω_3 periodic function, that is, it is left invariant by τ.

For the 51 unweighted walks associated to genus 1 curve and infinite group of the walk, only 9 walks give rise to a differentially algebraic generating function (see [16]). For these walks, it seems that the results of [3] show that there exists $g \in C_E$ such that $r_x(\omega) - g(\omega)$ is ω_3-periodic.

This line of research has great potential, and there remain many natural questions.

Open problem 6. For weighted walks associated to genus 1 curve and infinite group of the walk, is it always true that when $r_x(\omega)$ is differentially algebraic then there exists $g \in C_E$ such that $r_x(\omega) - g(\omega)$ is ω_3-periodic?

Open problem 7. What can be dervied from this approach about the differential dependence in t for the 51 unweighted walks associated to genus 1 curve and infinite group of the walk?

5.2 Finite group of the walk

Let us consider a walk associated with an elliptic curve $E_t = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (the curve defined by the kernel equation) and let us assume that this walk has finite group. This is equivalent to the fact that an integer multiple of ω_3 above belongs to $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. One can ask the following questions:

Open problem 8. The prolongation of $r_x(\omega)$ into a meromorphic function over $\mathbb{C}$ as well as the functional equation (8) obtained in [25] for the 51 unweighted walks associated to genus 1 curve and infinite group of the walk, depend heavily on the fact that the group of the walk is infinite. Can one still obtain analogous results in the finite group case?

Open problem 9. For weighted walks associated to genus 1 curve and finite group of the walk, is there a general closed form for the generating series?

6 Random walk processes and the critical exponent

Random walk processes are a classic topic of probability. Processes conditioned never to leave the cone are connected to Harmonic functions and Random Matrix theory. The persistance probabilities of random walks are linked to statistical physics and population models.

Many results on the expected first exit time from a cone can be converted into asymptotic enumeration results. The course of Kilian Raschel provided many very clear connections. If the number $q(n)$ of walks of length n grows asymptotically like $\kappa\rho^{-n}n^{-\alpha}$ as n tends to infinity, then we call α the critical exponent. Probability results give straightforward access to information on the critical exponents. The classical exponents for 1 dimensional models are 0, 1/2 and 3/2 which depend on the drift. The simple walk in two dimensions is a far richer. This is modelled by the simple $\{N, E, S, W\}$ walks in an arbitrary angular sector θ. Varopoulos, and Denisov and Wachtel give that the exponent for the total number of walks is $\pi/2\theta$ and for excursions is $\pi/\theta + 1$. These results imply that the generating functions are not D-finite whenever θ is not a rational multiple of π.

The non simple models can be similarly analyzed, once the model is suitably transformed. This plan of attack was used with great success by Bostan, Raschel and Salvy who demonstrated via a case analysis that for all small step models, if the group of the step set was infinite, the excursion generating function is not D-finite [7].

Open problem 10. Is there a direct connection between the infiniteness of the group and the irrationality of critical exponent? If so, does this correspondence hold in higher dimnesions?

In fact, the results of Denisov and Wachtel are for arbitrary dimension, so in theory a similar approach should work in higher dimension. The first model under consideration is the model known as the 3D Kreweras model, with step set $\{(1, 1, 1), (-1, 0, 0), (0, -1, 0), (0, 0, -1)\}$ restricted to the orthant $\mathbb{Z}^3_{\geq 0}$. Several groups [1, 22] have tried to estimate the critical exponent with a goal of establishing its irrationality.

Open problem 11. Is the generating function for the 3D Kreweras model D-finite?

7 Extensions of the original problem

7.1 Non convex regions

Non convex regions require a modified analysis. The 3/4 plane is the next candidate for a systematic study. Bousquet-Mélou has launched this for the simple steps [13], and has demonstrated that, while more technical, many of the same basic strategies can apply.

The most recent advances on this problem were presented by Trotignon during the meeting. Raschel and Trotignon have expressed the generating function for simple walks in the 3/4 plane as the solution to a boundary value problem [29].

7.2 Winding numbers

The majority of enumerative works on lattice walks to date use a last step recursive decomposition: A walk is either empty, or it is a walk plus a step. Timothy Budd described a very promising alternative decomposition [14]. He considers models with step sets that have a high amount of symmetry, and cones whose boundaries on the angles that are integer multiples of $\pi/4$. He considers a new decomposition for walks as a sequence of excursions. The work offers explicit formulas for 2D walks in a cone which avoid a boundary except at the start, and at the end. He considers the generating function where length and winding angle of the walk are tracked. The winding angle is the angle formed by the first and last step. This set up is sufficient to determine an explicit formula for the usual Gessel walks in the first quadrant. He gives some clear criteria for algebraicity, and the D-finiteness of the models that he considers is easy to establish.

This work raises a number of questions, and his intermediary objects may be useful to determine the combinatorics behind the classification of the generating functions.

Open problem 12. Can his argument deduce the generating function for simple walks in the 3/4 plane tracking winding number? How does it compare with the work of Raschel and Trotignon?

8 New Collaborations

Many participants have reported new collaborations as a result of this workshop, as per the intended goal. Several participants have launched collaborations to extend, and address many of the questions that were listed here. Several participants noted the quality of the open questions presented, and the applicability of techniques across domains.

Of particular note is an entirely new seminar series that was created at the historic Institut Henri Poincar (IHP): The *Groupe de travail autour des marches dans le quart de plan*[1] is organized by Lucia Di Vizio and Alin Bostan.

Michel Drmota and Andrew Rechnitzer have begun discussions on analytic properties of solutions of systems of functional equations.

Thomas Dreyfus reported several new collaborations, including new work on the algebraic independence of Mahler functions with Jason Bell, Boris Adamscewski and Michael Singer; and a new collaboration with Carlos Arreche on the elliptic hypergeometric functions.

Mark Wilson noted a boost to several projects in progress with Stephen Melczer.

Miklos Bona started a new collaboration with Jay Pantone at Dartmouth college on the log-convexity of certain permutation classes. They have used lattice paths to settle one of the special cases. Some of these classes are counted by generating functions that are not algebraic, but differentiably finite, and Miklos Bona is trying to use what he learned from the talk of Bruno Salvy about such functions.

Jason Bell and Marni Mishna considered some of the questions on the cogrowth problem raised by Igor Pak. Their work resulted in a preprint [2].

Michael Singer and Charlotte Hardouin have made progress towards Open Problem 6.

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