# NIKOL'SKII INEQUALITIES AND THEIR APPLICATIONS

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During our stay in Banff, we have studied the following questions:

- Remez and Nikol'skii's inequalities for trigonometric polynomials and spherical harmonics.
- Marcinkiewicz type inequalities for finite dimensional vector spaces of continuous functions.

Before discussing the obtained results, let us give a brief overview of the research field.

## 1 Overview of the Field

In the recent paper [4], a close connection between Remez and Nikol'skii's inequalities was shown. Here we restrict ourselves only to the questions related to Remez's inequalities.

Let $X_N$ denote a linear subspace of functions in $L^p()$ with $0 < p \le \infty$ and  being a probability space. The general form of the Remez inequality for a function $f \in X_N \subset L_p(\Omega)$, $0 < p \le \infty$, reads as follows: for any measurable $B \subset \Omega$ with measure $|B| \le b < 1$,

$$\|f\|_{L_p(\Omega)} \le C(N, |B|, p) \|f\|_{L_p(\Omega \setminus B)}.$$

Applications of Remez type inequalities include many different results in approximation theory and harmonic analysis; see [4] for more details and references.

For trigonometric polynomials $\mathcal{T}(Q)$ with frequencies from $Q \subset \mathbb{Z}^d$ (here $X_N = \mathcal{T}(Q)$ and $\Omega = \mathbb{T}^d$) the following result is well known [4]: For $d \ge 1$ and

$$Q = \Pi(\mathbf{N}) := \{\mathbf{k} \in \mathbb{Z}^d : |k_j| \le N_j, \quad j = 1, \dots, d\},$$

where $N_j \in \mathbb{N}^d$, for any $p \in (0, \infty]$, we have that $C(N, |B|, p) = C(d, p)$ provided that

$$|B| \le \frac{C}{\prod_{j=1}^d N_j}.$$

The investigation of the Remez-type inequalities for the hyperbolic cross trigonometric polynomials with

$$Q = \left\{ \mathbf{k} \in \mathbb{Z}^d : \quad \prod_{j=1}^d \max\{|k_j|, 1\} \le N \right\} \tag{1}$$ qq

has been recently initiated in [4]. It turns out that for such polynomials the problem of establishing the optimal Remez inequalities has different solutions when $p < \infty$ and $p = \infty$. If $p < \infty$, then

$$C(N, |B|, p) = C(d, p)$$

provided that

$$|B| \le \frac{C}{N}.$$

The known result in the case $p = \infty$ reads as follows.

T3.1 **Theorem 1.1.** *[4] There exist two positive constants $C_1(d)$ and $C_2(d)$ such that for any set $B \subset \mathbb{T}^d$ of normalized measure*

$$|B| \le \frac{C_2(d)}{N(\log N)^{d-1}}$$

*and for any $f \in \mathcal{T}(Q)$, where $Q$ is given by (1), we have*

$$\|f\|_\infty \le C_1(d)(\log N)^{d-1} \sup_{\mathbf{u}\in\mathbb{T}^d\setminus B} |f(\mathbf{u})|. \tag{2}$$ qqq

It is worth mentioning that this result is sharp with respect to the logarithmic factor. This is because the following statement is false.

*There exist $\delta > 0$, $A$, $c$, and $C$ such that for any $f \in \mathcal{T}(N)$ and any set $B \subset \mathbb{T}^d$ of measure $|B| \le (cN(\log N)^A)^{-1}$ the Remez-type inequality holds*

$$\|f\|_\infty \le C(\log N)^{(d-1)(1-\delta)} \sup_{\mathbf{u}\in\mathbb{T}^d\setminus B} |f(\mathbf{u})|.$$

# 2 Scientific Progress Made

We obtain a nontrivial Remez inequality for the hyperbolic cross trigonometric polynomials with no logarithmic factor in (2).

thm-remez **Theorem 2.1.** *For $d \ge 2$, let*

$$\alpha_d = \sum_{j=1}^{d} \frac{1}{j} \qquad \text{and} \qquad \beta_d = d - \alpha(d).$$

*There exist two positive constants $C_1(d)$ and $C_2(d)$ such that for any set $B \subset \mathbb{T}^d$ of normalized measure*

$$|B| \le \frac{C_2(d)}{N^{\alpha(d)}(\log N)^{\beta(d)}}$$

*and for any $f \in \mathcal{T}(Q)$, where $Q$ is given by (1), we have*

$$\|f\|_\infty \le C_1(d) \sup_{\mathbf{u}\in\mathbb{T}^d\setminus B} |f(\mathbf{u})|.$$

The proof is based on the the discretization inequality and the following Marcinkiewicz-type inequality for the hyperbolic cross polynomials. The latter result is interesting by itself and it reads as follows.

thm-2-1 **Theorem 2.2.** *For $d \ge 1$, let*

$$\alpha(d) = \sum_{j=1}^{d} \frac{1}{j} \qquad \text{and} \qquad \beta_d = d - \alpha(d).$$

*There exists a set $\Lambda$ of at most $C_d N^{\alpha(d)}(\log N)^{\beta(d)}$ points in $[0, 2\pi)^d$ such that for all $f \in \mathcal{T}(N)$,*

$$\|f\|_\infty \sim \max_{\omega\in\Lambda} |f(\omega)|.$$

Note that Theorem 2.2 for $d = 1$ is well known. More about Marcinkiewicz-type discretization can be found in [1]-[3].

# 3 Outcome of the Meeting

Currently we ( F. Dai, A. Prymak, S. Tikhonov, and V.N. Temlyakov) are working on the Marcinkiewicz-type discretization problems. We have made significant progress on this problem during our discussions in Banff. We are currently preparing for a joint paper, which will include several interesting results in this direction. We believe that the meeting served as an excellent start for the new collaboration on various problems in approximation theory related to polynomial inequalities and discretizations.

# References


VT158 [1] V.N. Temlyakov, The Marcinkiewicz-type discretization theorems for the hyperbolic cross polynomials, *Jaen Journal on Approximation* **9** (2017), No. 1; arXiv: 1702.01617v2 [math.NA] 26 May 2017.

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