Extensions of index theory inspired by quantum scattering theory (17frg668)

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1 Overview of the Field

In this section we describe the background to the meeting as it stood prior to the results announced by the participants.

The part of quantum theory that is relevant to this proposal is scattering theory. We approach scattering theory via the spectral shift function theory of Krein, Koplienko, and Potapov–Skripka–Sukochev [12, 13, 14, 15].

On the geometric side following from [6] the work of Carey, Grosse and Kaad has introduced a new spectral invariant the 'homological index' [9, 7]. This homological index provides the appropriate framework for a major extension of older ideas of R. Carey and J. Pincus [10] on almost commuting operators. It is expressed as a pairing of a new cyclic homology theory developed in [9] with its dual cohomology theory. An open question is whether we can relate the homological index to the spectral shift function. In addition we want to know if it is possible to use cyclic theory to investigate the latter and conversely use the spectral shift function to calculate the homological index.

One major obstacle is the lack of computable examples. We expect to be able to understand these spectral functions in the case of scattering for supersymmetric quantum Hamiltonians, Dirac-type operators on non-compact manifolds and their perturbations. In particular, our aim is to connect the spectral properties of the operators under investigation to invariants such as spectral flow, the Fredholm index, Witten index and generalisations. Previous work in this direction [1] uses the de Rham complex viewed à la Witten as a supersymmetric system. This work, though it shows connections to spectral geometry, is incomplete and the conjectures made there about extensions to more general situations were never published.

The work on the homological index in examples [7] shows that there is a deeper extension of index theory to the non-Fredholm case not understood in the 80s. In addition, our previous work has already led to a remarkable new insight into a question first raised by Atiyah-Patodi-Singer on the relationship of spectral flow for paths of Dirac operators on odd-dimensional manifolds to the Fredholm index for a Dirac operator on the suspension of the manifold. We summarise this aspect now because during the BIRS meeting progress was made.

This line of research started in [2] where it was shown that, when both exist, spectral flow is given by Krein's spectral shift function (even in semi-finite von Neumann algebras). Related work for operators with essential spectrum appeared in [11] in 2011 (a long paper in Advances in Math.) Two main results were obtained in this latter paper, the first being a trace formula and the second, a generalised Pushnitski formula [15].

The focus then however, was on Fredholm theory inspired by the fact that spectral flow relates directly to the topology of the underlying manifold on which the path of operators is defined. The surprising discovery made in [8] is that the trace formula due originally to Pushnitiski [15] and generalised in [11] is in fact an operator trace identity that has as a corollary, in the Fredholm case, the relationship between spectral flow and the Fredholm index first discovered by Atiyah-Patodi-Singer. The results of [8] do not depend on a Fredholm assumption and thus go far beyond the results of Robbin and Salamon [16].

The main aim of this focussed research group proposal is to investigate generalisations of [11] to situations involving non-Fredholm operators inspired by the main result of [8]. One specific question is whether the spectral shift function can be really thought of as a generalisation of spectral flow to non-Fredholm situations. As an adjunct to this we will investigate the Witten index as a method of calculating this generalised spectral flow.

In [11] and [6] it is assumed that we have self adjoint operators D and perturbations A such that for complex z not in the spectrum of D the product $A(D-z)^{-1}$ is trace class (i.e., one employs a relatively trace class condition). This rules out geometric examples. In work underway we have shown that the relatively Schatten class condition such as, $A(D-z)^{-n}$, $n = 2, 3, \dots$, is trace class, is what is needed for the geometric case (and PDEs in general).

Our primary technical objective is to relax this (severe) relatively trace class perturbation restriction so as to have very widespread applications in mathematical physics and geometry. There are some papers on the arxiv already on graphene and some unpublished work of Moore and Witten that contain examples of great interest. Moreover condensed matter theorists have speculated about generalisations of the Robbin-Salaman result for the study of spectral flow in their models.

Thus the study of examples in two and three dimensions is already important and it will assist to provide insight into general theory and applications. We anticipate interesting connections with condensed matter theory where spectral flow has played a role in the study of many model Hamiltonians.

2 Recent Developments and Open Problems

2.1 The spectral shift function. For a pair of self-adjoint not necessarily bounded operators A_0 and A_1 on a Hilbert space $\mathcal{H}$ such that their difference, $A_1 - A_0$, is trace class, there exists a unique function $\xi \in L^1(\mathbb{R})$ (known as the Krein–Lifshitz spectral shift function) satisfying the fundamental trace formula, $\mathrm{Tr}(\phi(A_1) - \phi(A_0)) = \int_{\mathbb{R}} \xi(x)\phi'(x)dx$, whenever ϕ belongs to a class of suitably admissible functions.

More precisely, we are considering the following situation. Take a path of self adjoint operators $A(t)$, $t \in \mathbb{R}$ on a Hilbert space $\mathcal{H}_0$ and form $\mathbf{D_A} = (d/dt) + \mathbf{A}$ acting on $\mathcal{H}_+ = L^2(\mathbb{R}; dt; \mathcal{H}_0)$. Here $(\mathbf{D_A}u)(t) = u'(t) + A(t)u(t) \quad u \in L^2(\mathbb{R}; dt; \mathcal{H}_0), \quad t \in \mathbb{R}$, and $\mathbf{A}$ is a direct integral of the family $\{A(t)\}_{t\in\mathbb{R}}$ (with asymptotes $A(t) \underset{t\to\pm\infty}{\longrightarrow} A_\pm$ in the norm resolvent sense) over $\mathbb{R}$. If we let $\mathcal{D}_+ = \mathbf{D_A}$, $\mathcal{D}_- = \mathcal{D}_+^*$ and $\mathcal{D}$ be the 2×2 matrix-valued operator introduced above acting on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_+$, then under a relatively trace class assumption on $A_\pm$ we can compute the relationship between spectral flow for the path $A(t)$, the spectral shift function for the pair $A_\pm$.

It is important to note that the spectral shift function also exists when no Fredholm condition is assumed and hence no standard notion of spectral flow exists. This raises the interesting question of what the spectral shift function is measuring geometrically when the Fredholm condition is absent.

2.2 Supersymmetry and graded spaces

We are given a Hilbert space $\mathcal{H}$ equipped with a $\mathbb{Z}_2$ grading γ, (that is, $\gamma^* = \gamma, \gamma^2 = I$), and a self adjoint unbounded operator $\mathcal{D}$ such that $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$. Then we may write,

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}.$$

In addition, we are given self-adjoint operators H_0 and H_1 (think of $H_0 = \mathcal{D}_+\mathcal{D}_-$ and $H_1 = \mathcal{D}_-\mathcal{D}_+$) acting on a separable graded Hilbert space $\mathcal{H}$, and ask for which scalar functions f, the difference

$f(H_1) - f(H_0)$ lies in the n^{th} Schatten–von Neumann ideal $\mathcal{L}^n(\mathcal{H})$. This question has been topical in perturbation theory for over 60 years starting in 1953 for the case where the difference $V := H_1 - H_0$ belongs to the trace class. In fact, Krein proved that

$$\xi(\lambda; H_1, H_0) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \Im\left(\ln(D_{H_1/H_0}(\lambda + i\varepsilon))\right) \text{ for a.e. } \lambda \in \mathbb{R}, \tag{1}$$

or, equivalently, in terms of the perturbation determinant,

$$\det((H_0 - zI)(H_1 - zI)^{-1}) = \exp\left(\int_{\mathbb{R}} \frac{\xi_{H_0,H_1}(\lambda)}{(\lambda - z)} d\lambda \right), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{2}$$

Based on what we know about the relationship between Krein's spectral shift function and standard spectral flow, we have conjectured that there is a generalised spectral flow formula, for a relatively Schatten class perturbation condition in the non-Fredholm case.

2.4 Details on the higher Schatten conjectures.

Recalling the notation of subsection 2.2, we remark that Atiyah, Patodi, and Singer studied operators of the form $\mathbf{D_A}$ that we defined there. This class of operators form models for more complex situations. They arise in connection with Dirac-type operators (on compact and noncompact manifolds), the Maslov index, Morse theory (index), Floer homology, Sturm oscillation theory, etc.

One of the principal aims of such a study is to compute the Fredholm index of $\mathbf{D_A}$. The most general published computation of this kind to date is contained in [11] under the assumption that $A_+ - A_-$ is of trace class relative to A_-. Note that prior to [11] and [15] it was required to assume that operators in the path $A(t)$ had discrete spectrum.

Invoking the spectral shift function $\xi(\lambda; A_+, A_-)$ and the perturbation determinant associated to the pair (A_+, A_-), $D_{A_+/A_-}(z) = \det_{\mathcal{H}}((A_+ - zI)(A_- - zI)^{-1})$ corresponding to the pair (A_+, A_-), the following was established in [11],

$$\begin{aligned} \operatorname{Index}(\mathbf{D_A}) &= \operatorname{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty}) = \xi(0; A_+, A_-) \\ &= \pi^{-1} \lim_{\varepsilon \downarrow 0} \Im\left(\ln\left(\det_{\mathcal{H}}\left((A_+ - i\varepsilon)(A_- - i\varepsilon)^{-1} \right) \right) \right) \\ &= \xi(0_+; H_1, H_0). \end{aligned}$$

We conjectured that these results will generalise when we have relatively Schatten class perturbations.

Work underway

We have a complete theory for the case of one dimension that has just been published. With $A_- = \frac{d}{idt}$, $A_+ = A_- + F$ where F is a self adjoint $N \times N$ matrix acting on $\mathcal{H} = L^2(\mathbb{R}; \mathbb{C}^N)$, we were able to establish the following:
(i) A Pushnitksi formula (cf. [15]), implying that

$$\xi_{A_-,A_+} = \xi_{H_1,H_2},$$

where H_1, H_2 are as above acting on $\mathcal{K} = L^2(\mathbb{R}, \mathcal{H})$;
(ii) explicit formulas for ξ_{A_-,A_+} for these examples;
(iii) the trace formula:

$$2z\operatorname{Tr}_{\mathcal{K}}[(z + H_1)^{-1} - (z + H_2)^{-1}] = \operatorname{Tr}_{\mathcal{H}}[g_z(A_-) - g_z(A_+)],$$

where $g_z(x) = x(z + x^2)^{-1/2}$ (a smoothed version of the sign-function for $z < 0$);
(iv) an abstract framework for relatively Hilbert-Schmidt perturbations $A_+ - A_-$ in which the Pushnitski formula and hence the result mentioned in (i) could be proved;
(v) the existence of a class of examples of pseudo-differential operators in any dimension to which the method developed in [11] applies.

The main gap

To utilise the methods we have employed previously, we need to study scattering for Dirac-type operators in two and three dimensions. The existing literature is not adequate for our purposes. The objective then is to extend the one-dimensional analysis described above point by point to higher dimensions.

3 Presentation Highlights

(i) Alan Carey summarized the state of the art of spectral flow as it pertained to our meeting.

(ii) Galina Levitina explained the proof of the principal trace formula and the generalisations of Pushnitski's work, applicable to higher dimensions.

(iii) Roger Nichols and Fritz Gesztesy described in detail the proof of absence of singular continuous spectrum for flat space, massless Dirac-type operators based on the strong limiting absorption principle in all space dimensions ≥ 2. In addition, a global limiting absorption principle (for the entire real line) for free, massless Dirac-type operators, again in all space dimensions ≥ 2, was presented.

(iv) Jens Kaad explained the recent work on the homological index and the resolvent version of the principal trace formula (PTF).

4 Scientific Progress Made

As a result of Galina Levitina's presentation we can announce an end point to the work begun in [11] so that the relationship between the Witten index and generalised spectral and between the associated spectral shift functions has been completely settled [5].

In interactions between the participants, the outstanding example of two-dimensional systems using Dirac-type massless Hamiltonians was resolved in the affirmative. In fact, a first attempt prior to our Banff meeting required the limitation of space dimension ≥ 3, but due to interactions at the meeting this could be extended to the important two-dimensional case (with potential interest in connection with graphene applications). Thus, a proof of the global limiting absorption principle in all space dimensions ≥ 2 emerged during the meeting and will result in paper [4]. The latter also contains essential self-adjointness results of Dirac-type operators which are of interest in their own right. No doubt, without the opportunity afforded by BIRS, this research would not have been done as it involved essential contributions by nearly every participant at the meeting.

In addition, a proof of the limiting absorption principle for all nonzero energies for massless Dirac-type operators with matrix-valued potentials (applicable to electromagnetic potentials decaying at infinity like $|x|^{-\rho}$, with $\rho > 1$) in all dimensions ≥ 2, has now been established [3].

A plan was formulated for publications from the meeting and we have identified at least three lengthy papers (or even monographs), reporting our results being completed over the next six months.

5 Outcome of the Meeting

The principal outcomes were,

(i) the finalisation of the proof of the principle trace formula as an operator trace identity (cf. [5]),

and,

(ii) establishing the limiting absorption principle for certain Dirac-type operators on flat space in all dimensions ≥ 2 (cf. [3], [4]).

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