New Trends in Graph Coloring

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1 Overview of the Field

Graph coloring is one of the oldest studied topics in graph theory. Its roots date back to 1852 with the first statement of the celebrated Four Color Conjecture: can the countries of any map on a globe be colored with at most four colors so that no two countries that share a common boundary have the same color? It took over 100 years to prove this conjecture, and the attempts to do so gave rise to many other important concepts in graph theory and motivated the study of graph colorings in greater generality. Further motivation for the concept comes from wide-ranging applications of many variants of graph coloring in algorithm design, scheduling and resource allocation.

For these reasons, the study of graph colorings is a very active research subject with many interesting open questions. While many of these questions are notoriously difficult, a plethora of new and innovative methods have been developed in graph coloring over the last decade.

Let us mention some of the most promising directions that were explored during the workshop.

1.1 Density of critical graphs

A graph is $(k+1)$-*critical* if its chromatic number is $k+1$, but every proper subgraph is k-colorable. Every graph that is not k-colorable contains a $(k+1)$-critical subgraph, which would make their description important in many graph coloring problems. Of course, a nice exact description is unlikely to exist in general, however, the situation is more promising when restricted to sparse graphs such as graphs embedded in a fixed surface.

Indeed, Thomassen [39] proved that for any $k \geq 5$, there are only finitely many $(k+1)$-critical graphs embedded in any fixed surface. This in particular implies that k-colorability of such graphs can be decided in polynomial time. This result was later strengthened by giving explicit bounds on the sizes of $(k+1)$-critical embedded graphs, even in the list coloring setting [35], which has further consequences such as much improved bounds on their edgewidth, number of distinct k-colorings, or precoloring extension properties. A similar theory for 3-colorability of embedded triangle-free graphs was developed by Dvořák, Král and Thomas, leading to results such as resolving a conjecture of Havel regarding 3-colorability of planar graphs with triangles far apart [13].

The results mentioned so far fundamentally rely on the properties of embedded graphs. Rather surprisingly, Kostochka and Yancey [32] recently gave a much more general result showing that for some of these problems, only the density of the graph matters. Their lower bound on density of critical graphs almost completely resolved a longstanding question of Ore. The bound is based on an interesting new idea (called *potential method*) that has since been a basis of a number of results both for ordinary proper graph coloring and its variants; for example, Dvořák and Postle [15] were able to apply it in the circular coloring setting. Furthermore, using this method it is possible to obtain best known bounds on the sizes of 4-critical graphs of girth five, as well as critical graphs with bounded clique number.

1.2 New probabilistic tools

Probabilistic methods found many applications in graph coloring theory, both in constructions of graphs with large chromatic number and as tools to find proper colorings. Let us in particular mention the celebrated construction of Erdős [18] demonstrating graphs with arbitrarily large girth and chromatic number, as well as Johansson's bound [26] on the chromatic number of triangle-free graphs of bounded maximum degree.

The probabilistic tools that make it possible to show a lower bound on the probability of the conjunction of many almost independent events are of great interest in study of chromatic properties of bounded degree graphs. The basic such tool is Lovász Local Lemma. Based on a new constructive proof of Lovász Local Lemma, Gonçalves, M. Montassier, and A. Pinlou [23] formulated a new technique to show existence of colorings, called *entropy compression method*. The idea of this method is to analyze a random process that might eventually result in a proper coloring based on a record of its run, and to show that the number of such records of runs that fail is eventually dominated by the number of random choices in the run of a given length. Many bounds previously proved using Lovász Local Lemma can be significantly improved using this technique. Bernshteyn [4] provides an analysis of this method that makes it possible to reframe it in an entirely combinatorial fashion (without having to design and analyze the algorithm), stated as Local Cut Lemma.

1.3 χ-boundedness

Chromatic number of a graph is lower bounded by its clique number, but other than that there is no relation in general; indeed, there exist triangle-free graphs with arbitrarily large chromatic number [18]. On the other hand, the class of *perfect graphs* (whose chromatic number is equal to their clique number, hereditarily) has received much attention, especially given the recent proof of Strong Perfect Graph Theorem characterizing these graphs.

It is natural to ask about graph classes in that the chromatic number is bounded by a function of their clique number, called *χ-bounded* graph classes. In his paper introducing the concept, Gyarfás [25] raised a number of questions about χ-bounded graph classes that till recently defied all attempts to solve them. This was changed in a series of break-through results of Seymour and Scott [36, 37].

Although we now have a large body of work regarding χ-bounded graph classes and their properties, there is a little of what would amount to a coherent theory of the concept of χ-boundedness, and even in the cases where χ-boundedness of a class has been established, the quantitative relationship between the chromatic number and clique number is poorly understood (e.g., we have no examples of classes where a superpolynomial lower bound would be proved, but no lack of classes where only singly or doubly exponential upper bounds are known). The results of Seymour and Scott renewed the interest in the concept and bring some hope of rectifying the situation.

1.4 Old problems approached via new variants of colorings

Occasionally, a problem that appears hard can be easier to solve by first generalizing and then using extra tools available in the generalized setting. This might be the case for some of list coloring problems; here, a *list coloring* is a proper coloring where each vertex is restricted in its choice of colors to a list assigned to it. Many techniques for ordinary proper coloring do not translate to this setting, especially if they involve identification of vertices (which may not be possible for vertices with different lists).

To overcome this difficulty for a particular list coloring problem, Dvořák and Postle [16] introduced the notion of *correspondence coloring* where assigning a color to a vertex may prevent usage of different colors at the neighbors; this makes some transformations that do not work in the list coloring setting possible, of course at the expense of potentially making the considered problem harder. Other applications were found since, and the correspondence coloring seems to be interesting as a concept of its own as well.

2 Workshop Programme and Presentation Highlights

We took the opportunity given by presence of some of the most renowned researchers in the area and asked them to give in-depth tutorials on some of the methods and hot topics we mentioned in the introduction.

- Paul Seymour, Alex Scott, and Ingo Schiermeyer contributed a two-part tutorial on the recently very active topic of χ-boundedness.

  Paul Seymour provided an in-depth survey on the known results in the area as well as the basic techniques used to obtain them, culminating in an overview of the ideas of their recent results with Scott on graphs without odd holes and without long holes.

  Alex Scott continued with a detailed explanation of more difficult results for graphs excluding a tree of radius two, and for graphs excluding a subdivision of a given tree.

  Ingo Schiermeyer briefly discussed the few classes for that somewhat sharp bounds on the function bounding the chromatic number with respect to the clique number are known.

- Luke Postle and Matthew Yancey gave a detailed tutorial on the application of the potential method for coloring sparse graphs, which they co-developed in the last few years.

  Matthew Yancey described step-by-step the process of finding a proof by this method and demonstrated it in a detail on the problem of finding a decomposition of a sparse graph into an independent set and a forest.

  Luke Postle surveyed some of the more advanced results obtained using this technique, especially regarding the density of critical graphs without cliques of a given size, as well as applications to other kinds of coloring such as the circular coloring.

Many of presentations at the workshop also focused on recently developed methods and notions.

- Alexander Kostochka gave a survey talk on the properties and applications of correspondence coloring. As we mentioned in the introduction, correspondence coloring was originally developed to deal with list coloring problems, and many of the properties mentioned in the talk either mirror those known for list coloring. Nevertheless, he also highlighted many differences, such as a quantitatively different behavior with respect to the average degree—average degree d forces the list chromatic number to be at least $\Omega(\log d)$, while the correspondence chromatic number is much larger, at the order of $\Omega(d/\log d)$.

- David Wood gave talk showcasing several methods used to solve problems related to non-repetitive colorings. The arguments included the use of probabilistic tools such as Lovász Local Lemma and its recently developed strengthening—the entropy compression method. He also gave examples of use of tree decompositions not restricted by width, but by other properties of the bags.

- Anton Bernshteyn gave a short tutorial on the usage of Local Cut Lemma, a strengthening of the entropy compression method, providing a step-by-step demonstration on the problem of density of critical hypergraphs.

- Paul Wollan gave a talk outlining several approaches to coloring graphs avoiding a fixed graph as an immersion, including the usage of Kempe chains to serve as edges of the immersed graph.

- Louis Esperet's talk showed some methods for coloring geometric graphs and the connections to bounding the integrality gap for the maximum packing of cycles in planar digraphs.

- Daniel Cranston's talk gave an introduction into methods used for edge-coloring problems, including a detailed argument showing the properties of Kierstead paths, and their far-reaching generalization into Tashkinov trees (both of these are basic tools that enable one to extend an edge-coloring of a subgraph to an edge coloring of the whole graph). He also described an application of these tools to coloring linegraphs of multigraphs.

Several other presentations focused on particular problems and results of interest:

- Zdeněk Dvořák's talk outlined the equivalence of Thomassen's conjecture on exponentially many 3-colorings of planar triangle-free graph to new conjecture on satisfiability of a constant fraction of prescribed restrictions for such 3-colorings. The latter problem seems of independent interest in other settings.
- Marthe Bonamy presented a lower bound for the algorithmic complexity of multicolorings (relative to Exponential Time Hypothesis), showing an interesting way how a SAT instance can be succinctly transformed into a set coloring instance.
- Chun-Hung Liu gave talk on partitions of planar graphs without certain cycle lengths. He showed a complete characterizations of excluded cycle lengths necessary to force a planar graph to have a partition into an independent set and a graph of bounded maximum degree.
- Daniel Král' presented several results on colorings of planar graphs—a progress on colorings of 3-connected graphs so that vertices incident with a common face receive different colors, and a construction of a surprising counterexample to a long-standing conjecture of Steinberg.
- Robert Šámal presented a result on 3-flows, showing that a 3-edge-connected graph has a 3-flow that is only zero on at most one sixth of the edges.

3 Open Problems

We solicited among the participants open problems that they consider to be of interest. We believe these collected problems give an insight into the current trends and developing topics in the area, and we are considering the publication of the collection.

The problems range from new and potentially easy to solve ones to long-standing open questions and their variations. They cover a variety of subjects, from traditional topics such as the chromatic properties of planar graphs, to several new and potentially influential notions of coloring. The topic of χ-bounded graph classes is touched on by several of the problems, reflecting recent interest and progress in the area.

Let us mention some of the problems here.

3.1 Louis Esperet: Polynomial bounds on chromatic number in hereditary χ-bounded classes

Problem: *Is it true that for every hereditary χ-bounded class $\mathcal{G}$, there is a constant C such that $\chi(G) \leq \omega(G)^C$ for all graphs $G \in \mathcal{G}$?*

Let us recall that a class of graphs $\mathcal{G}$ is *χ-bounded* if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$. Thus, χ-bounded classes are "almost perfect".

This problem outlines the difficulty in proving exact functions that bound the chromatic number in such classes—while we now have fairly developed tools for proving χ-boundedness qualitatively, the provided bounds look wasteful and are likely to be far from optimal. On the other hand, finding graphs with large chromatic number subject to structural restrictions is notoriously hard.

A concrete case where the polynomial lower bound/exponential upper bound gap waits to be closed for a long time concerns *circle graphs* (intersection graphs of chords of a circle). The circle graphs are known to be colorable using $21 \cdot 2^{\omega}$ colors (ern [10], improving previous results [30, 31]), and Kostochka [30] constructed circle graphs requiring $\Omega(\omega \log \omega)$ colors.

Let us remark that there exist hereditary classes whose χ-bounding function is at least a polynomial of an arbitrarily large degree, namely the classes of graphs without induced stars $K_{1,k}$. Let G be a graph with $R(k, \omega+1)-1$ vertices that contains neither an independent set of order k nor a clique of order greater than ω. Then G does not contain an induced star $K_{1,k}$ and its chromatic number is at least $|V(G)|/(k-1) = \Omega(\omega^{k/2})$ by the lower bound on the off-diagonal Ramsey numbers of [5]. On the other hand, if a graph G with clique number ω does not contain $K_{1,k}$, then its maximum degree is less than (and the chromatic number is at most) $R(k, \omega)$, and thus these classes cannot give a counterexample to the considered problem.

A positive answer to the problem would have the following interesting consequence. Let H be a graph such that the class $\mathcal{G}$ of graphs avoiding H as an induced subgraph is χ-bounded. In that case every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq \omega(G)^C$, and thus $\alpha(G) \geq |V(G)|/\chi(G) \geq |V(G)|/\omega(G)^C$, and $\max(\alpha(G), \omega(G)) \geq |V(G)|^{1/(C+1)}$. In other words, every such graph H would have the Erdős-Hajnal property. Let us remark that there are many graphs whose exclusion is known to imply χ-boundedness (paths, trees of radius 2, ...), but for which the Erdős-Hajnal conjecture is open. In the light of this, it seems more likely that the answer to the question is negative.

3.2 Nicolas Trotignon: Polynomial χ-bounds

Esperet asks whether every hereditary χ-bounded class has a χ-bounding function that is a polynomial. This question is certainly extremely difficult. Here are two potentially easier questions.

First, let us point out one interesting special case of Esperet's problem.

Problem: *For any integer $k \geq 1$, is the class of graphs of rank-width at most k χ-bounded with a polynomial dependence of the chromatic number on the clique number?*

Dvořák and Král' [12] proved that for any integer k, graphs of rank-width at most k form a χ-bounded class.

For the second question, we need the following definition. For any hereditary class of graphs $\mathcal{G}$, let $f_{\mathcal{G}}$ be the *optimal χ-bounding function for $\mathcal{G}$* defined by

$$f_{\mathcal{G}}(x) = \max\{\chi(G) \text{ for all graph } G \in \mathcal{G} \text{ such that } \omega(G) = x\}.$$

A general question is the following.

Problem: *For a given non-decreasing function f, is there a hereditary class $\mathcal{G}$ such that $f_{\mathcal{G}} = f$?*

For instance, is there a class such that $f_{\mathcal{G}}(2) = 2$, and $f_{\mathcal{G}}(x)$ is huge for $x > 2$? The answer is no. Indeed, since $f_{\mathcal{G}}(2) = 2$, any graph in $\mathcal{G}$ has no odd hole. By a theorem by Scott and Seymour [36], the class of odd-hole-free graphs is χ-bounded. It follows that $f_{\mathcal{G}}$ cannot be too "huge".

More specific question is then the following:

Problem: *Given an integer k, is there a hereditary class of graphs $\mathcal{G}$ such that $f_{\mathcal{G}}(2) = 3$ and $f_{\mathcal{G}}(3) = k$?*

3.3 Marthe Bonamy: Edge deletions in graphs without induced cycles of length divisible by 3

Problem: *Let G be a non-empty graph without induced cycles of length divisible by* 3*. Does G necessarily contain an edge e such that $G - e$ also contains no induced cycles of length divisible by* 3*?*

Graphs without induced cycles of length divisible by 3 are known to have bounded chromatic number as proved by Bonamy et al. [6]. If the assertion of the problem holds, this would imply that such graphs are actually 3-colorable by an adaptation of an argument of Wrochna (who used it to prove that graphs without any cycles of length divisible by 3 are 3-colorable).

A far-reaching strengthening of [6] was recently proved by Scott and Seymour [37], whose result implies that for every $a > b \geq 0$, a triangle-free graph of large chromatic number contains an induced cycle of length $b \pmod{a}$.

Hence, it might seem that the choice of the class (avoiding induced cycles of length divisible by 3) is quite arbitrary; curiously though, the class seems to have some interesting properties. Aharoni and Haxell [1] point out one reason why this could be the case, based on the topological properties of the complex of independent sets of such graphs. For a graph G, let $I(G)$ denote the abstract simplicial complex formed by the independent sets of G. Kalai and Meshulam conjectured that each graph with a large chromatic number has an induced subgraph H such that the sum of Betti numbers of $I(H)$ is large. They also conjectured that if G has no induced cycles of length divisible by 3, then the sum of Betti numbers of $I(G)$ is at most 1. In the former conjecture, they also proposed the variant with the sum of Betti numbers replaced by Euler characteristic. This motivates the following conjecture proposed a few years ago.

Problem: *Show that if G is a graph without induced cycles of length divisible by 3, then the Euler characteristic of $I(G)$ is in absolute value at most 1. In other words, the numbers of odd and even independent sets of G differ by at most 1.*

3.4 Dan Král': Weak Steinberg's conjecture

Steinberg [38] conjectured that all planar graphs without cycles of lengths four or five are 3-colorable; however, this conjecture was recently disproved [11]. On the other hand, Borodin et al. [9] proved that it suffices to forbid cycles of lengths 4 to 7. This leaves open the following question.

Problem: *Is every planar graph containing no cycles of lengths four, five or six 3-colorable?*

The study of chromatic number of planar graphs without cycles of given lengths was started by Grtzsch [24] who proved that excluding triangles suffices to guarantee 3-colorability. Naturally, a question arises regarding the planar graphs with triangles restricted in some way (in full generality, 3-colorability of planar graphs is NP-complete to decide [22]). Planar graphs with at most three triangles are 3-colorable [2], and non-3-colorable planar graphs with exactly 4 triangles were fully described recently [7]. An approximate description of minimal non-3-colorable planar graphs with a bounded number of triangles can be gleaned from the theory of Dvok, Krl' and Thomas, which also implies that planar graphs with triangles far apart are 3-colorable [13].

The last mentioned result answers in positive an old question of Havel [27]. However, the provided bound on the required distance is rather large (on the order of 10^{100}), while the best know lower bound (given by Aksionov and Mel'nikov [3]) is 4. Thus, the following problem still stands.

Problem: *Determine the minimum integer d such that every planar graph with distance at least d among its triangles is 3-colorable.*

The best partial result towards resolving this theorem is by Borodin et al. [8], who proved that distance 4 is sufficient in planar graphs without 5-cycles that share at least one edge with a triangle.

3.5 Zdeněk Dvořák: Fractional chromatic number of planar graphs of girth at least 5

Problem: *Does there exist a constant $c < 3$ such that every planar graph of girth at least 5 has fractional chromatic number less than c?*

An n-vertex graph with fractional chromatic number c must contain an independent set of size at least c. Supporting the assertion of the problem, it is known [14] that there exists $c < 3$ such that all n-vertex planar graphs of girth at least 5 have independence number at least n/c.

The girth restriction in the problem cannot be relaxed, since Pirnazar and Ullman [34] found triangle-free planar graphs with fractional chromatic number arbitrarily close to 3 (on the other hand, Dvok et al. [17] proved that the fractional chromatic number of every planar triangle-free graph is strictly smaller than 3). Furthermore, we cannot replace fractional chromatic number by circular chromatic number in the statement of the problem, since for example odd wheels whose spokes are subdivided once have circular chromatic number exactly 3.

More generally, it is natural to ask what is the maximum fractional chromatic number of planar graphs of given girth. Without girth restriction, there exist planar graphs of fractional chromatic number exactly 4; describing planar graphs with smaller fractional chromatic number is an open question, related to another well-known open problem of recognizing n-vertex planar graphs that have independent set larger than $n/4$. A significant source of complications in both problems comes from the fact that the only known way how to prove the bound is using the Four Color Theorem.

3.6 Penny Haxell: Strong chromatic number of graphs with maximum degree 2

Problem: *Let G be a graph of maximum degree at most 2 (i.e., a disjoint union of paths, cycles, and isolated vertices) whose number of vertices is divisible by 4. Let G' be a graph obtained from G by partitioning its vertices into groups of size 4 and adding all edges within each such group. Is G' necessarily 4-colorable?*

The answer to the analogous question with 4 replaced by other integer k is true for $k \geq 5$ by the result of Haxell [28], and false for $k \leq 3$ by a construction of Fleischner and Stiebitz [21].

For an integer k, a *k-clique enlargement* of a graph G is a graph obtained from G by adding at most $k - 1$ isolated vertices to make its number of vertices divisible by k, partitioning the vertices into groups of size k, and adding all edges within each such group. We say that a graph G is *strongly k-colorable* if every k-clique enlargement of G is k-colorable. The *strong chromatic number* of G is the minimum k such that G is strongly k-colorable; a non-trivial fact that G is also strongly k'-colorable for every $k' \geq k$ was proved by Fellows [19].

One motivation for this notion comes from the well-known "cycle plus triangles" conjecture (proved by Fleischner and Stiebitz [20]), which can be stated as the claim that cycles of length divisible by 3 are strongly 3-colorable.

Haxell [28] proved a Brooks-like result for strong coloring: a graph of maximum degree Δ has strong chromatic number at most $3\Delta - 1$, improved later[29] to $\frac{11}{4}\Delta$. On the other hand, Fleischner and Stiebitz [21] constructed for each Δ an example of a Δ-regular graph with strong chromatic number at least 2Δ, and it has been conjectured that the corresponding upper bound holds as well. The presented problem thus concerns the first unresolved case of this conjecture, for $\Delta = 2$.

3.7 David R. Wood: Majority colorings of digraphs

A *majority coloring* of a digraph is a function that assigns each vertex v a color, such that at most half the out-neighbours of v receive the same color as v. In other words, more than half the out-neighbours of v receive a color different from v (hence the name 'majority'). Whether every digraph has a majority coloring with a bounded number of colors was posed as an open problem on mathoverflow [40]. In response, Ilya Bogdanov proved that a bounded number of colors suffice for tournaments [40]. Kreutzer et al. [33] solved the problem, showing that every digraph has a majority 4-coloring. They also found infinitely many digraphs with no majority 2-coloring (e.g. cyclically oriented K_3). The following problem naturally arises:

Problem: *Does every digraph have a majority 3-coloring?*

Or, indeed, the following natural generalization.

Problem *For an integer $c \geq 2$, does every digraph have a vertex $(2c - 1)$-coloring such that for each vertex v, at most $\frac{1}{c}\deg^+(v)$ out-neighbours of v receive the same color as v?*

This result would be best possible for all $c \geq 2$, as shown by the cyclic orientation of K_{2c-1}.

4 Scientific Progress Made

Both the open problems and the presentations lead to interesting discussions among the participants, inspiring new ideas and questions. Some new results were obtained already during the workshop and several cooperations on the other questions were started.

Paul Wollan asked about a variant of the problem presented in Subsection 3.3: Given a graph with no induced cycles whose length is a multiple of a fixed integer m, does there necessarily exist an edge whose removal preserves this property? Paul Seymour found a counterexample for $m = 5$, which he together with Sergey Norin and Marthe Bonamy generalized to all odd $m > 3$. Dan Cranston, Felix Joos and Marthe Bonamy found a counter-example for any m divisible by 4. A construction of Robert Šámal and Paul Seymour gives a counterexample for $m = 6$. It seems likely that there exist counterexamples for $m = 4k + 2$ with $k \geq 2$ as well, while the cases of $m = 2$ and $m = 3$ might have positive answers.

Problems regarding majority coloring (see Subsection 3.7) also attracted some attention. Gregory Gauthier gave an approach towards resolving the first problem of majority 3-coloring, and Fiachra Knox found an elegant argument that in the second problem, a $2c$-coloring with the required properties exists.

Motivated by a talk by Zdeněk Dvořák, Luke Postle proposed several questions regarding the possibility to satisfy a constant fraction of requests in a coloring; e.g., for which $k \geq 5$ does there exist $\varepsilon > 0$ such that given an n-vertex planar graph, an assignment L of lists of size k, and a function r assigning to each vertex v a color $r(v) \in L(v)$, is it possible to find an L-coloring of the graph that matches r on at least εn vertices? Joined also by Sergey Norin, they showed that the answer is affirmative for $k \geq 6$ and found interesting results for several other related problems. They are preparing a short note introducing this concept, establishing its basic properties and presenting their results.

Alexandr Kostochka and Anton Bernshteyn continued their discussion of correspondence chromatic number, proving that there is $f(n) \leq 5n^2/4$ such that for every n-vertex graph G, the graph $G^\star$ obtained from G by adding $f(n)$ dominating vertices has DP-chromatic number equal to $\chi(G)$. This answers the corresponding question of Xuding Zhu. On the other hand, they found an example that needs approximately $n^2/4$ extra vertices to have this property. It follows that the Noel-Reed-Wu Theorem for list coloring extends to correspondence coloring only in a very weak form. Also, they found an example of a planar bipartite graph with correspondence chromatic number 4.

With Pavol Hell, Penny Haxell and Luke Postle, they also talked about a generalization of the concept to homomorphisms, and identified a nice question about its complexity. They also discussed this question later in Bordeaux at the Bordeaux Graph Workshop and will continue thinking about.

Paul Wollan and David Wood started a collaboration regarding the problems related to the topic of David's talk (nonrepetitive chromatic number), and are now close to completing a paper entitled "Nonrepetitive colourings of graphs excluding a fixed immersion".

Lena Yuditsky, Maria Aksenovich, Ingo Schiermeyer, and Marthe Bonamy started a cooperation on improving the χ-bounding function for the family of $2K_2$ free graphs, which was one of the open problem that Ingo Schiermeyer suggested for the workshop. Initially, they obtained a modest additive improvement by 2 to the currently known bounds.

Chun-Hung Liu started a joint work with Louis Esperet for proving or disproving that every triangle-free planar graph of bounded maximum degree can be two colored in a way that every monochromatic component has bounded size, reflecting both his presentation at the workshop and one of the problems raised by Louis Esperet. He also started a cooperation with Bojan Mohar and Hehui Wu regarding the problem of finding an orientation of a d-degenerated graph with small maximum out-degree such that no pair of adjacent vertices have the same out-degree (inspired by a problem suggested by Bojan Mohar for the workshop).

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