Lefschetz Properties and Artinian Algebras

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1 Overview of the Field

Both weak and strong Lefschetz properties can be defined for an endomorphism of a finite graded vector space over a field. Thus the definition can be understood by even beginning graduate students of mathematics, and the basics of the theory of Lefschetz properties could become part of the tool kit of all mathematicians. As such the theory does not belong to a particular field of mathematics.

The strong Lefschetz property (in the narrow sense) can be interpreted as a representation of the Lie algebra $sl(2)$. The Clebsch–Gordan decomposition of modules over $sl(2)$ is the foundation of this theory. It was discovered by a physicist although it is purely mathematical in nature. In quantum mechanics it is used for the "addition of angular momentum". It provides the irreducible decomposition of the tensor product of $SO(3)$-modules. (The Lie algebras $so(3)$ and $su(2)$ are isomorphic and $su(2)$ and $sl(2)$ have the same types of irreducible representations.)

The origin of the strong Lefschetz property is the Hard Lefschetz Theorem for the cohomology rings of projective non-singular algebraic varieties over the complex number field (or more generally, compact complex manifolds). The homology and cohomology rings can be computed in many ways; using homology of topological spaces, singular cohomology, deRham cohomology, sheaf cohomology, etc. It is an amazing fact that they all result in the same cohomology rings.

In an unexpected application, it was used by R. Stanley in 1980 to prove that certain posets have the Sperner property [33]. For commutative algebraists the Hard Lefschetz Theorem of cohomology rings was a powerful tool to compute the Rees number of Artinian rings. The Clebsch–Gordan decomposition can be used to prove the flat extension theorem which has developed into a powerful tool for the study of the strong Lefschetz property. The strong Lefschetz property of monomial complete intersections is an immediate corollary. (It leaves us with a philosophical question: why is $sl(2)$ so powerful?)

Springer Lecture Notes 2080, "The Lefschetz Properties", published in September 2013, is probably the first book on this subject [13]. It is written from the viewpoint of commutative algebra, but, unlike the traditional treatment of Artinian rings, the authors regard Artinian rings as cohomology rings of some (probably non-existent) algebraic varieties. In the last chapter of this book the authors show an interesting application of the Schur-Weyl duality theorem.

The timely appearance of the book "On the Geometry of Some Special Projective Varieties" by Francesco Russo [28] coincident with our WorkShop, provides an addition to the meagre literature on this subject. It was awarded a prize from the Italian Mathematical Union. The author makes a reference to the Lefschetz properties as "an unexpected appearance of hypersurfaces with vanishing Hessian."

As we said in the proposal for this workshop, a systematic approach to the theory has just begun, and no one knows the true nature of this theory. We have worked together only for 3–4 years, but during that time have reached the conclusion that the theory of Lefschetz properties has a promising future.

2 Recent Developments and Open Problems

2.1 Rodrigo Gondim

We recall some basic facts on Gordan-Noether theory of forms with vanishing Hessian.

Proposition 1. **(Gordan-Noether criterion)** *Let $f \in \mathbb{K}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. Then $\operatorname{Hess}_f = 0$ if and only if the partial derivatives of f are algebraically dependent.*

Definition 2. $R = \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]$, for $m \geq 3$ and $n \geq 2$ be the polynomial ring and Q be the associated ring of differential operators. A Perazzo polynomial of degree d is a reduced polynomial $f \in R$ of the form

$$f = x_1 g_1 + \ldots + x_n g_n + h \tag{1}$$

where $g_i \in \mathbb{K}[u_1, \ldots, u_m]_{d-1}$ for $i = 1, \ldots, n$ are algebraically dependent but linearly independent and $h \in \mathbb{K}[u_1, \ldots, u_m]_d$. The algebra $A = Q/\operatorname{Ann}(f)$ is called a Perazzo algebra and f a GNP form.

For the Hessians of order $k \geq 2$ we do not have a criterion for their vanishing. We only have a sufficient condition that is essentially equivalent to the existence of a huge zero block in the Hessian matrix.

Proposition 3. *Let $R = \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]$ be a polynomial ring in $N + 1 = m + n + 1$ variables, let Q be the associated ring of differentials and for $f \in R_d$ let $A = A(f) = Q/\operatorname{Ann}(f)$. Set $\deg(f) = d = e + k$, with $e > k \geq 1$ and assume that $\operatorname{Ann}(f)_1 = 0$. Consider also $\tilde{R} = \mathbb{K}[u_1, \ldots, u_m]$, $\tilde{Q}$ and $B = \tilde{Q}/\operatorname{Ann}(f) \cap \tilde{Q}$. Suppose that $\alpha_1, \ldots, \alpha_s \in A_k \setminus B_k$ are linearly independent differential operators such that $f_{\alpha_1}, \ldots, f_{\alpha_s} \in \mathbb{K}[u_1, \ldots, u_m]_e$. If $s > \binom{m+k-1}{k}$, then*

$$\operatorname{Hess}_f^k = 0.$$

Consider the GNP hypersurfaces given by monomials $N_1, \ldots, N_s \in \mathbb{K}[x_1, \ldots, x_n]_k$ and $M_1, \ldots, M_s \in K[u_1, \ldots, u_m]_e$ using all the variables.

Lemma 4. *Let $f = \sum_{i=1}^{s} N_i M_i$. If $s > \binom{m-1+k}{k}$, then*

$$\operatorname{Hess}_f^k = 0.$$

The Higher Hessians give us a criterion to fail the SLP. Using mixed Hessians we get a similar criterion for the WLP.

Definition 5. Let $A = Q/\operatorname{Ann}_Q(f) = \bigoplus_{k=0}^{d} A_k$ be a standard graded Artinian Gorenstein $\mathbb{K}$ algebra of socle degree d. Let $i \leq j \leq \frac{d}{2}$ be two integers and let $B_k = \{\alpha_1, \ldots, \alpha_s\}$ and $B_l = \{\beta_1, \ldots, \beta_t\}$ be bases of the K-vector spaces A_k and A_l respectively. The (mixed) Hessian matrix of f of order (k, l) is the matrix:

$$\operatorname{Hess}_f^{(k,l)} = (\alpha_i(\beta_j(f)))_{s \times t}.$$

We denote $\operatorname{Hess}_f^k := \operatorname{Hess}_f^{(k,k)}$, $\operatorname{Hess}_f^k := \det(\operatorname{Hess}_f^k)$ and $\operatorname{Hess}_f := \operatorname{Hess}_f^1$.

The following result is a generalization of a Theorem due to Maeno - Watanabe.

Theorem 6. (Hessian Lefschetz criterion)

Let $A = Q/\operatorname{Ann}(f)$ be a standard graded Artinian Gorenstein algebra of codimension r and socle degree d and let $L = a_1x_1 + \ldots + a_rx_r \in A_1$, such that $f(a_1, \ldots, a_r) \neq 0$. The map $\bullet L^{l-k} : A_k \to A_l$, for $k < l \leq \frac{d}{2}$, has maximal rank if and only if the (mixed) Hessian matrix $\operatorname{Hess}_f^{(k,d-l)}(a_1, \ldots, a_r)$ has maximal rank.

Let $N_1, \ldots, N_s \in \mathbb{K}[x_1, \ldots, x_n]_k$ and let $M_1, \ldots, M_s \in \mathbb{K}[u_1, \ldots, u_m]_e$ be monomials using all the variables. Consider the GNP form f.

Lemma 7. *Let $f = \sum_{i=1}^{s} N_iM_i$. If $s > \binom{m-1+e-1}{e-1}$, then*

$$\operatorname{Hess}_f^{(k,e-1)} = 0.$$

In other words, the map $\bullet L : A_k \to A_{k+1}$ is not injective for any $L \in A_1$.

Problem 8. Criterion for the vanishing of Higher Hessians *Can we find a criterion for the vanishing of Higher Hessians, analogous to the GN Proposition 1, for the vanishing of the usual Hessians?*

Problem 9. Low codimension *In the paper [4] M. Boij et al reduce the problem to compressed algebras. Can we give a characterization of a form $f \in R = \mathbb{K}[x_1, x_2, x_3]$ such that the associated algebra $A = Q/\operatorname{Ann}(f)$ is compressed? Are we able to calculate their mixed Hessians and its rank? It could be a strategy to prove the WLP for codimension 3.*

Problem 10. Complete intersections *Given a complete intersection $I \subset Q$ how should we find the Macaulay dual generator, that is $f \in R$ such that $I = \operatorname{Ann}(f)$? If we were able to do this, then it should be possible to use the Hessian Lefschetz criterion to prove that all CI have the SLP.*

1. Theoretically, we can calculate f from I using Macaulay duality.
2. Algorithmically?
3. Explicit formulae for special cases?

Problem 11. *For GNP forms we have some partial results about the annihilator ideal, Hilbert functions, Lefschetz properties, Jordan blocks. It should be interesting to answer these questions in general for GNP.*

2.2 Migliore

Problem 12. *If C is an irreducible, arithmetically Cohen-Macaulay curve in $\mathbb{P}^4$ then does the general hyperplane or hypersurface section of C have the property that a general Artinian reduction has WLP?*

Problem 13. *If C is an irreducible, non-arithmetically Cohen-Macaulay curve in $\mathbb{P}^4$ whose Hartshorne-Rao module has the "expected" behavior with respect to the multiplication from any component to the next by a general linear form, then does the general hyperplane or hypersurface section of C have the property that a general Artinian reduction has WLP?*

2.3 Murai

Problem 14. *For any balanced Gorenstein simplicial complex Δ, the canonical Artinian reduction of $K[\Delta]$ has the SLP in characteristic 0.*

See Conjecture 18 below for more details.

3 Presentation Highlights

- **Mats Boij** gave the opening survey talk “The Weak Lefschetz Property and Betti Tables” on March 14.

  The work of Juan Migliore and Uwe Nagel [23] gives upper bounds on the graded Betti numbers of Artinian Gorenstein algebras satisfying the WLP. Idealization gives examples of Artinian Gorenstein algebras failing the WLP and where the graded Betti numbers exceed these bounds. However, the first such examples we encouter are in codimension nine and it is unknown whether there are such examples in codimension four. In codimension three the structure theorem by D. Buchsbaum and D. Eisenbud implies that the bounds are always satisfied, while it is still open whether all Gorenstein algebras in codimension three satisfy the WLP. While the examples mentioned above might suggest that Gorenstein algebras failing the WLP have large Betti numbers, there are also many examples where such algebras have the minimal possible Betti numbers given the Hilbert function.

  The conclusion is that the relation between the WLP and Betti tables is unclear and needs to be studied further.

- **David Cook II:** David Cook II gave the talk ”Large Lefschetz defects” on 15 March 2016 based on joint work with U. Nagel. The talk expanded upon the connection between the failure of the weak Lefschetz property and the existence of an associated projective variety satisfying Laplace equations described in the earlier talk by E. Mezzetti. The magnitude of the failure, which also counts the number of Laplace equations the variety satisfies, is called the Lefschetz defect. In the talk, he provided a naive upper bound for the Lefschetz defect and constructed, for the case of three variables, an ideal that achieves this bound asymptotically.

- **Rodrigo Gondim:** We introduce a family of standard bigraded binomial Artinian Gorenstein algebras, whose combinatorial structure characterizes the ones presented by quadrics. These algebras provide, for all socle degrees greater than two and in sufficiently large codimension with respect to the socle degree, counter-examples to the Migliore-Nagel conjectures [23]. In particular there are families of counterexamples to the conjecture that Artinian Gorenstein algebras presented by quadrics should satisfy the weak Lefschetz property. We also prove a generalization of a Hessian criterion for the Lefschetz properties given by Watanabe, which is our main tool to control the Weak Lefschetz property. This is joint work with G. Zappalá [11].

- **Tony Iarrobino:** Suppose the Artinian Gorenstein algebra A of vector space dimension n over a field k is a quotient $A = R/I$ of a regular local ring R by an ideal I. An element ℓ in the maximum ideal of R determines a nilpotent multiplication map m_ℓ on A, and a partition P_ℓ of n, the *Jordan type* of the pair (ℓ, A). The set of such Jordan types, and in particular the Jordan type for a generic ℓ are invariants of A. The strong Lefschetz property for the pair requires A to have unimodal Hilbert function H and P_ℓ is the conjugate $H^\vee$ to H. The pair is weak Lefschetz if the number of parts of P_ℓ is the maximum value of H. We give examples of A for which P_ℓ for a generic linear form ℓ is strictly in-between the least Jordan type consistent with weak Lefschetz and the Jordan type $H^\vee$ for a strong Lefschetz element. We propose the far more general problem of classifying Artinian Gorenstein algebras by P_ℓ for a generic $\ell \in \mathfrak{m}$ and give examples.

- **Leila Khatami:** The Jordan type of a nilpotent matrix is the partition giving the sizes of the Jordan blocks in the normal Jordan form of the matrix. In this talk we discuss all partitions that have a fixed partition Q as the generic Jordan type in their nilpotent commutator. We report on a joint work with A. Iarrobino, B. Van Steirteghem and R. Zhao in which we provide a complete description of all such partitions for a partition Q with at most two parts [16]. In particular we arrange all such partitions in a table that we denote by $\mathcal{T}(Q)$. We then report on an ongoing joint project with M. Boij, A. Iarrobino, B. Van Steirteghem and R. Zhao in which we study the equations of loci in $\mathcal{T}(Q)$.

- **Chris McDaniel:** Chris McDaniel gave a survey talk about the “rather bold” conjecture posed by Junzo Watanabe at the Lefschetz Workshop held at Göttingen in March 2015:

  Junzo’s Bold Conjecture. *Every standard graded Artinian complete intersection algebra can be embedded in some standard graded Artinian complete intersection algebra generated by quadrics with the same socle degree.*

In his talk, Chris reported on progress made on this conjecture in collaboration with Larry Smith– specifically that Junzo's bold conjecture holds for coinvariant rings of pseudo-reflection groups generated by reflections of order two. In this case, the quadratic complete intersection algebras are the so-called Bott-Samelson algebras associated with pseudo-reflection groups.

Chris also talked about the deep structural results on Bott-Samelson algebras associated with Coxeter groups due to W. Soergel [29] and the subsequent recent work of B. Elias and G. Williamson which established the famous long-standing Kazhdan-Lusztig positivity conjecture [9]. Related to Junzo's bold conjecture, one can ask if the embedded complete intersection algebra inherits the strong Lefschetz property from the quadratic one in which it is embedded. In his talk, Chris pointed out that one can use the aforementioned results of Soergel and Elias-Williamson to show that a coinvariant ring does not always inherit the strong Lefschetz property from the Bott-Samelson algebra in which it is embedded. Finally he computed an example showing that this type of behavior already occurs for Coxeter groups in type A_3.

Chris further mentioned that, besides coinvariant rings, Junzo's bold conjecture also holds for monomial complete intersections and, more generally, complete intersections whose polynomial generators split into products of linear forms [20].

- **Emilia Mezzetti:** Emilia Mezzetti gave a survey talk about Togliatti systems and Artinian ideals failing weak Lefschetz property. She first recalled the relationship, due to apolarity, between homogeneous Artinian ideals I of the polynomial ring over an algebraically closed field of characteristic 0 which fail the Weak Lefschetz Property - WLP - and projective varieties X satisfying at least one Laplace equation of order s, i.e. such that all the s-osculating spaces have dimension strictly less than expected. A first consequence of this connection was the classification of the smooth toric rational threefolds parametrized by cubics, satisfying a Laplace equation of order 2, extending a classical theorem of E. Togliatti for surfaces [35]. This gave a different point of view on the example of H. Brenner and A. Kaid [6]. This also led to a conjecture, successively proved by M. Michalek and R.M. Miró-Roig, about ideals generated by cubic monomials in any number of variables. She then surveyed some recent results about Artinian ideals of the polynomial ring in any number of variables generated by monomials of any degree d and failing the WLP in degree $d-1$. These are also called Togliatti systems. Since the picture soon becomes much more involved than in the case of cubics, it is natural to restrict attention to Togliatti systems that are minimal and smooth, addressing the question of their minimal and maximal number of generators. This question has been solved and the classification of the systems with minimal number of generators, or number of generators close to the minimal, has been achieved. The proofs involve combinatorial methods of toric geometry. This was joint work with Giorgio Ottaviani and Rosa Maria Miro-Roig [22], and with Rosa Maria Miro-Roig [21].

- **Juan Migliore:** Juan Migliore gave the talk "Geometric Aspects of the WLP" on March 16, 2016. The motivating question is whether there are natural geometric conditions on finite sets of points so that the general Artinian reduction has the Weak Lefschetz Property. An important open question is whether it is true for every reduced, arithmetically Gorenstein set of points, but this is known to be a difficult problem. After this, the most natural question is whether it is true when the points have the Uniform Position Property. In an old paper with Jeaman Ahn, Migliore showed that even this is not true by drawing a connection with certain non-arithmetically Cohen-Macaulay curves in projective space and their Hartshorne-Rao modules [2]. In this talk he recalled basic facts about these modules, described a counterexample from that paper, and proposed new problems jointly with Uwe Nagel and Hal Schenck.

- **Uwe Nagel:** Uwe Nagel gave the talk "Unexpected curves, line arrangements, and Lefschetz properties". Based on joint work with D. Cook II, B. Harbourne, and J. Migliore [7] connections between Lefschetz properties and the study of Hilbert functions of (fat) points as well as the theory of line arrangements were discussed. Consider a finite set Z of points in the plane with the property that, for some integer j, the dimension of the linear system of plane curves of degree $j+1$ through the points of Z and having multiplicity j at a general point is unexpectedly large. Criteria for the occurrence of such unexpected curves were given and the range of their degrees was described. Inspired by work of R. Di Gennaro, G. Ilardi, and J. Vallès, we related properties of Z to properties of the arrangement of lines dual to the points of Z. In particular, we got a new interpretation of the splitting type of a line arrangement, and we showed that the existence of an unexpected curve is equivalent to the failure of a certain Lefschetz property. This implies a Lefschetz-like criterion for H. Terao's conjecture on the freeness of line arrangements.

- **Eran Nevo:** We consider two problems on Betti tables of monomial ideals generated in degree 2 (edge ideals, after polarization): 1. Strand connectivity: can rows in the Betti table have internal zeros? [A. Conca, G. Whieldon] 2. Subadditivity: for t_i the maximal j for which $\beta_{i,j}$ is nonzero, must $t_{a+b} \leq t_a + t_b$? (J. Herzog-H. Srinivasan L. Avramov-A. Conca-S. Iyengar [3, 15]). We show that for the first question the answer is NO for the first 2 rows and YES otherwise. We use it in showing that for the second question the answer is YES for $b = 1, 2, 3$ (for $b = 1$ this was proved by J. Herzog-H. Srinivasan, for all monomial ideals). Via Hochster formula, our proofs are topological-combinatorial. Joint work with A. Abedelfatah [1].

- **Hal Schenck** gave a survey talk on BGG:

  The Bernstein-Gelfand- Gelfand correspondence is an equivalence between certain derived categories over the polynomial algebra S and the exterior algebra E. In down to earth terms, it allows one to define a functor from S-modules to complexes of free E-modules with linear differential, and vice versa. This connects to the multiplication map used to investigate Lefschetz properties, but it seems little has been done to explore this. This talk gave an outline of first steps in this direction, beginning with the definition of a Koszul algebra A and the quadratic dual $A^!$. Then the BGG correspondence was outlined, with several concrete examples worked out. The talk concluded with a definition of the Tate resolution and examples of how to compute the Betti table of a graded S-module from the BGG machinery.

- **Larry Smith:** Fix a ground field F and denote by $R = F[V]$ the graded algebra of homogeneous polynomial functions on $V = F^n$. Write R^g for the subalgebra of R pointwise fixed by an element $g \in GL_n(F)$. For $g_1, g_2, ..., g_k \in GL_n(F)$, following Bott and Samelson, we introduce the algebra $BS(g_1, g_2, ..., g_k) = R \otimes_{R^{g_1}} R \otimes_{R^{g_2}} \cdots R \otimes_{R^{g_k}} R$ which we call the Bott-Samelson algebra and an Artinian reduction $\overline{BS}(g_1, g_2, ..., g_k) = F \otimes_R BS(g_1, g_2, ..., g_k)$ of it which we call the reduced Bott-Samelson algebra. We prove that for a k -tuple of reflections $s_1, s_2, ..., s_k$ the reduced Bott-Samelson algebra is a complete intersection algebra and, if F has characteristic zero, then $BS(s_1, s_2, ..., s_k)$ has the strong Lefschetz property. If $\rho : G \to GL_n(F)$ is a faithful reflection representation of a finite group G we show how to construct an embedding of the coinvariant algebra $F[V]_G$ into a reduced Bott-Samelson algebra $\overline{BS}(s_1, s_2, ..., s_k)$ where $s_1, s_2, ..., s_k$ are reflections generating G, provided that the coinvariant algebra is fixed point free, meaning that the fixed point set of the group G acting on its coinvariant algebra consists of the scalar multiples of the identity element alone. In the nonmodular case, i.e., the case where the order $|G|$ of G is prime to the characteristic of the ground field, coinvariant algebras are always fixed point free, we construct such a Bott-Samelson embedding $F[V]_G \hookrightarrow \overline{BS}(s_1, s_2, ..., s_k)$ which is a degree one map between Poincaré duality algebras. In certain favorable cases, e.g., if all the reflections in G have order 2, which is the case for all but three of the primitive complex reflection groups of degree at least 3, as was proven by H.F. Blichfeldt, we deduce by the Subring Theorem, that $F[V]_G$ has the strong Lefschetz property provided the field F has characteristic zero. For the three exceptions we have a game plan to show that they too have coinvariant algebras with the SLP.

- **Jean Vallès:** Jean Vallès spoke about special singular hypersurfaces characterizing the failure of SLP of some Artinian rings. In a recent paper, E. Mezzetti, R.M. Miró-Roig and G. Ottaviani highlight the link between rational varieties satisfying a Laplace equation and Artinian ideals failing the Weak Lefschetz Property [21]. In a joint work with R. di Gennaro and G. Ilardi we extended this link to the more general situation of Artinian ideals failing the Strong Lefschetz Property and characterize the failure of the SLP (which includes WLP) by the existence of special singular hypersurfaces (cones for WLP) [10]. Thank to this characterization Jean Vallès related the splitting type of derivation bundles associated to a line arrangements to the failure of the SLP of some Artinian ideals and gave new examples of ideals failing the SLP.

- **Adela Vraciu:** We give a characterization, depending on the characteristic $p > 0$ of the field k, for the values $d_1, ..., d_n$ (under certain triangle-inequality-like restrictions) for the quotient of a polynomial ring in n variables by the powers of the variables raised to powers $d_1, ..., d_n$ to have WLP (see [36]).

- **Junzo Watanabe:** Junzo Watanabe gave a talk on “The EGH conjecture and the Sperner property of complete intersections.” He defined the matching property of Gorenstein algebras, and then he showed three interesting consequences of the property.

  (1) Matching property of a Gorenstein algebra implies the Sperner property.

  (2) If the EGH conjecture is true, then the Sperner property holds on all complete intersections independent of the characteristic.

  (3) If a complete intersection is defined by products of linear forms, it has the Sperner property.

  (Joint work with T. Harima and A. Wachi [14]).

4 Scientific Progress Made

4.1 Boij, Gondim, Migliore, Nagel, Seceleanu, Schenck, Watanabe

The idealization construction by M. Nagata has been used in many cases to construct Artinian Gorenstein algebras failing the WLP, for example the famous example by R. Stanley. In terms of Macaulay inverse systems, we can construct the idealization of the canonical module of an Artinian level algebra with inverse system $\langle F_1, F_2, \ldots, F_s\rangle$ as the Gorenstein algebra with inverse system $\langle \sum x_i F_i\rangle$ where $x_1, x_2, \ldots, x_s$ are new variables. Here the forms $F_1, \ldots, F_s$ are assumed to be homogeneous of degree d in variables $y_1, y_2, \ldots, y_n$. We studied the generalized construction given by the inverse system $\langle \sum x_i^e F_i\rangle$, where $e \geq 1$.

The conclusions of our studies of the Gorenstein algebra given by $\langle \sum x_i^e F_i\rangle$ were that

- For $e \geq d$ it satisfies the WLP for any forms F_i.
- For $e < d$ it can fail to have to satisfy the WLP and in many cases this can be seen directly from the bigraded Hilbert function.
- For $e = d$ it seems to satisfy the SLP in many cases and in fact, the higher Hessians that determine the SLP are polynomials with all of their coefficients of the same sign in many cases over the rational numbers.

For example when $F_0, F_1, \ldots, F_9$ are the monomials of degree 3 in y_1, y_2, y_3 we get the Hessian

$$
\begin{aligned}
&-2^7 3^{10} 5(27x_0^3x_1^3x_2^3y_1^6 + 12x_1^6x_2^3y_1^5y_2 + 72x_0^3x_2^3x_3^3y_1^5y_2 + 27x_0^3x_1^3x_4^3y_1^5y_2\\
&\qquad\qquad\vdots\\
&\qquad + 27x_4^3x_8^3x_9^3y_2y_3^5 + 27x_5^3x_8^3x_9^3y_3^6)(y_1y_2y_3)^9\prod_{i=0}^{9}x_i
\end{aligned}
$$

where the coefficients are positive integers adding up to $2 \cdot 5^3 7^2$. In order to check that the SLP holds it is sufficient to find one non-zero term in this Hessian. It would be interesting to study the positivity of the coefficients of these Hessians.

The proof of the WLP for $e \geq d$ relies on the fact that the linear form $\ell = \sum x_i$ is a Lefschetz element and by duality we only have to check injectivity up to degree d.

4.2 Mezzetti, Miró-Roig, Vallès, Iarrobino, Khatami,

We studied the geometric properties of certain Artinian ideals failing Weak Lefschetz Property (WLP, for short) More concretely, we have studied the theorem of Eugenio Togliatti that gives the characterization of the only rational surface X contained in the n-dimensional projective space, with $n \geq 5$ parametrized by cubics such that the osculating space at a general point of X has dimension less than 5, and whose apolar ideal I is Artinian. The proof we have studied is due to Jean Vallès following an idea of Laurent Gruson, and is inspired by the original proof given by Togliatti in 1946 [35]. The main idea is that the morphism from $\mathbb{P}^2$ to $\mathbb{P}^3$ defined by the linear system of the cubics in I must be a $3:1$ cyclic Galois covering of a surface S. We have observed that there is a whole class of examples of minimal monomial Togliatti systems I_d in the polynomial ring $S = K[x, y, z]$ (with char $K = 0$), for any degree d, generalizing Togliatti example, defining a $d : 1$ cyclic Galois covering f of the image surface. If we write d in the form $d = 2k$, resp. $2k + 1$, the number of generators of I_d is $k + 3$. We have studied the geometry of this family of examples. We have seen that they produce interesting arrangements of lines in $\mathbb{P}^2$. We studied the Togliatti systems of plane curves defining a cyclic Galois covering with the aim of giving a complete classification. This is connected with the description of the monomials which are invariant under the action of the cyclic group of order d, embedded in $GL(3, C)$, for the various possible embeddings. We have also tried to understand the possible connections with the decomposition in Jordan blocks of the endomorphism of S/I defined as multiplication by the linear form L.

Finally, we studied the osculatory behaviour of the Togliatti surface and of some other classes of Togliatti systems of plane curves, trying to understand in which points the appropriate osculating space has dimension less than in the general point. In the case of the Togliatti systems called trivial in the article by E. Mezzetti and R. M. Miró-Roig [**?**] the situation is similar to the one for Castelnuovo surfaces described by A. Lanteri and R. Mallavibarrena in a recent paper [19]

4.3 Cook II, Faridi, Juhnke-Kubitzke, Murai, Nevo, Vraciu

Our group studied the following problem about the Strong Lefschetz Property (SLP, for short) of Artinian reductions of Stanley–Reisner rings of simplicial polytopes.

We say that a simplicial d-polytope P with the vertex set V is **balanced** if its graph is d-colorable, in other words, if there is a map $c : V \to \{1, \dots, d\}$ such that $c(u) \neq c(v)$ whenever u and v form an edge of P. Let $K[P]$ be the Stanley–Reisner ring of P with coefficients in an infinite field K. It was proved by Stanley [32] that if a simplicial d-polytope is balanced, then a sequence of linear forms $\theta_1, \dots, \theta_d$, defined by $\theta_i = \sum_{v \in V,\ c(v)=i} x_v$ for $i = 1, \dots, d$, is a system of parameters of $K[P]$, where x_v is the variable corresponding to the vertex $v \in V$. Such a system of parameters is uniquely determined by P since the choice of a coloring map $c : V \to \{1, \dots, d\}$ is unique (up to permutations of the colors), and called the **canonical system of parameters** for $K[P]$. For a balanced simplicial d-polytope P and its canonical system of parameters $\theta_1, \dots, \theta_d$, we call the Artinian reduction $K[P]/(\theta_1, \dots, \theta_d)$ **the canonical Artinian reduction of** $K[P]$.

Our group worked on the following problem.

Problem 15. *For any balanced* 3*-polytope* P*, prove that the canonical Artinian reduction of* $K[P]$ *has the SLP in characteristic* 0*.*

The above canonical Artinian reduction is an Artinian Gorenstein algebra which naturally appears in algebraic combinatorics, and it is tempting to prove the SLP for the canonical Artinian reduction of $K[P]$ for any balanced d-polytope. However, since this looks to be a hard problem, we studied the case when $d = 3$ as a first step.

We succeeded in getting an affirmative answer to Problem 15. More precisely, we proved

Theorem 16. *For any balanced* 3*-polytope* P*, the canonical Artinian reduction of* $K[P]$ *has the SLP if* $\mathrm{char}(K)$ *is not* 2 *or* 3*.*

We note that the assertion is false in characteristics 2 and 3. Indeed, the canonical Artinian reduction of the Stanley–Reisner ring of a cross 3-polytope is isomorphic to $K[x, y, z]/(x^2, y^2, z^2)$ which fails to have the SLP in characteristics 2 and 3.

The proof we found is an interesting combination of algebra and combinatorics. To study this problem, we regard simplicial d-polytopes as simplicial $(d - 1)$-spheres (namely, simplicial complexes which are homeomorphic to the $(d - 1)$-sphere), and consider the following two combinatorial operations. Note that, when we discuss combinatorial types, considering simplicial 2-spheres and considering simplicial 3-polytopes do not make any difference.

Connected sum. If a simplicial 2-spheres Δ_1 and Δ_2 intersect in a single triangle σ, the simplicial 2-sphere

$$\Delta_1 \#_\sigma \Delta_2 = (\Delta_1 \setminus \{\sigma\}) \cup (\Delta_2 \setminus \{\sigma\})$$

is called the **connected sum** of Δ_1 and Δ_2.
Balanced contraction. Let Δ be a balanced simplicial 2-sphere and u, v the vertices having the same color. Let $\mathrm{st}_\Delta(w) = \{\sigma \in \Delta : \{w\} \cup \sigma \in \Delta\}$ be the star of a vertex w in Δ. If $\mathrm{st}_\Delta(v) \cup \mathrm{st}_\Delta(u)$ is a 2-dimensional disc and $\mathrm{st}_\Delta(v) \cap \mathrm{st}_\Delta(u)$ consists of two edges, then we define the simplicial 2-sphere

$$\Delta' = (\Delta \setminus (\mathrm{st}_\Delta(v) \cup \mathrm{st}_\Delta(u))) \cup \{u \cup \sigma : \sigma \in \partial(\mathrm{st}_\Delta(v) \cup \mathrm{st}_\Delta(u))\}.$$

We call the operation $\Delta \to \Delta'$ a **balanced contraction**.

Then we proved Theorem 16 by the following argument: We say that a balanced 2-sphere Δ has the colored SLP if the canonical Artinian reduction of $K[\Delta]$ has the SLP. First, we prove that if Δ_1 and Δ_2 have the colored SLP, then so does its connected sum. Second, we prove that, in a contraction $\Delta \to \Delta'$, the colored SLP of Δ' implies the colored SLP of Δ. Finally, we show that every balanced 2-sphere is obtained from the boundary complexes of a regular octahedron by taking connected sums and by taking the inverse of the contraction operation repeatedly. From these, we get the desired SLP since for the regular octahedron the corresponding Artinian reduction is $K[x, y, z]/(x^2, y^2, z^2)$ which has the SLP if $\mathrm{char}(K)$ is not 2 or 3.

We found one more interesting result on the Lefschetz property of colored 3-polytopes. A simplicial 2-sphere Δ is said to be $(2, 1)$**-balanced** if we can color the vertices of Δ with blue and red so that every facet of Δ has exactly two blue vertices and one red vertex. Stanley [32] proved that, for $(2, 1)$-balanced simplicial complex Δ, there is a system of parameters $\theta_1, \theta_2, \theta_3$ such that θ_1 and θ_2 are linear combination of blue vertices and θ_3 is a linear combination of red vertices. We call such a system of parameters, $(2, 1)$**-colored system of parameters**. Considering Theorem 16, it is natural to ask if a similar result holds for $(2, 1)$-balanced simplicial 2-spheres. Somewhat surprisingly, we get the following negative answer for this question: Let Q be a 3-polytope, F the number of facets of Q and V the number of the vertices of Q. Let Δ be the simplicial 2-sphere obtained from Q by subdividing all the facets of Q. Then Δ is $(2, 1)$-balanced by coloring the vertices of Q with blue and coloring other vertices with red.

Theorem 17. *With the same notation as above, for any* $(2, 1)$*-colored system of parameters* $\theta_1, \theta_2, \theta_3$ *of* $K[\Delta]$ *and a linear form* w*, the kernel of the multiplication map*

$$\times w : (K[P]/(\theta_1, \theta_2, \theta_3))_1 \to (K[P]/(\theta_1, \theta_2, \theta_3))_2$$

has at least dimension $F - V + 1$*. In particular, if* $F \geq V$ *then* $K[P]/(\theta_1, \theta_2, \theta_3)$ *fails to have the SLP.*

Possible Next Problem

By the result proved by our working group, it is tempting to conjecture the following. We say that a simplicial complex is Gorenstein if its Stanley–Reisner ring is Gorenstein.

Conjecture 18. *For any balanced Gorenstein simplicial complex* Δ*, the canonical Artinian reduction of* $K[\Delta]$ *has the SLP in characteristic* 0*.*

The conjecture is interesting for several reasons. First, the above Artinian reduction is an Artinian Gorenstein algebra which naturally appears in algebraic combinatorics. From an algebraic point of view, for such a naturally occurring object one has to ask about its possible Lefschetz properties. Another interesting aspect is that the canonical Artinian reduction has a nice multigraded structure induced by the coloring where as a generic Artinian reduction destroys the multigraded structure of $K[\Delta]$. This multigraded structure plus Lefschetz properties will have more applications to the study of Hilbert functions. Second, the balanced condition naturally appears in combinatorics. For example, balanced complexes contain important combinatorial objects such as order complexes and Coxeter complexes. It is known that the solution of the conjecture has interesting combinatorial consequences such as a generalization of the balanced Generalized lower bound inequality [18] for all balanced Gorenstein simplicial complexes. Third, this algebra also appears in toric topology and has a geometric meaning. Indeed, the ring is known to be the cohomology ring of a certain topological object. We expect that this might be interesting from a topological point of view.

We think that Conjecture 18 is a difficult problem, but the results proved by this working group gives a first step to solve this difficult conjecture.

4.4 McDaniel, Smith, Wachi

In the break-out problem sessions at BIRS, our group, consisting of Chris McDaniel, Larry Smith, and Akihito Wachi, focused on Lefschetz properties of coinvariant rings and their relationship with Bott-Samelson algebras. First, following Larry's suggestion, we proved a technical lemma which gives sufficient conditions for the base ring in a Leray-Hirsch type decomposition of the coinvariant ring to have the strong Lefschetz property. Larry was able to use this lemma to establish the strong Lefschetz property for coinvariant rings for the pseudo-reflection groups numbered 25 and 26 in the Shephard-Todd list.

Next, motivated by a "rather bold" conjecture posed by Junzo Watanabe at the previous Lefschetz workshop, our group considered the following problem:

Problem. *Prove that the coinvariant ring of a pseudo-reflection group embeds into some Bott-Samelson algebra with the same socle degree.*

Prior to this BIRS workshop, Chris and Larry had solved this problem for groups generated by pseudo-reflections of order 2 using a neat result of Larry and his co-authors, Frank Neumann and Mara Neusel, which computes the Hilbert ideal of a pseudo-reflection group from its Demazure operators. In an effort to extend these arguments to the higher order cases, our group attempted to reconstruct the Neumann-Neusel-Smith result using what we called "generalized Demazure operators". Specifically, we first established a generalized Leibniz formula for these operators, then we used this formula to show that the set of polynomials annihilated by certain compositions of our generalized Demazure operators is an ideal which contains the Hilbert ideal. We further conjectured that this containment is actually equality. Since this BIRS workshop, the three of us have continued to collaborate on this problem, and we hope to have a paper out soon.

4.5 Other working groups

In addition to the break-out problem sessions, some smaller groups got together to discuss related problems.

- Mats Boij, Juan Migliore, Rosa-Maria Miró-Roig and Uwe Nagel met on two evenings with the goal of discussing how to finish a long-standing (approx. 3 years) project and submit it for publication. We hope that it will be done in the next two months.
- Mats Boij, Tony Iarrobino, and Leila Khatami met several times, making progress using a tropical matrix tool to determine equations for loci of matrices of a given Jordan type P, in the commutator of Q, when Q has two parts and is the generic commuting Jordan type of P. Joint work with Bart van Steirteghem and Rui Zhao.
- Sara Faridi and Adela Vraciu met twice to investigate the Weak Lefschetz Property in bipartite graphs.
- Rodrigo Gondim and Junzo Watanabe met several times to discuss the Macaulay dual generator of binomial complete intersections and its 2nd Hessians in five variables.
- Rodrigo Gondim and Tony Iarrobino met to discuss examples studied by Gondim of Artinian Gorenstein algebras for which the generic Jordan type of a multiplication map is strictly between those for weak and strong Lefschetz.
- Juan Migliore, Uwe Nagel, and Hal Schenck met on three evenings with the goal of starting a new project on the WLP question for certain algebras presented by quadrics. We hope that the project has enough momentum at this point to keep moving forward even after we are no longer able to meet in person.

  Title "The Weak Lefschetz Property and quotients by quadratic monomial ideals"

  Abstract: M. Michałek–R.M. Miró-Roig recently gave a beautiful geometric characterization of quotients by ideals generated by quadratic or cubic monomials such that the multiplication map by a general linear form fails to be injective in the first nontrivial degree [22]. Their work was motivated by conjectures of G. Ilardi and E. Mezzetti-R.M. Miró-Roig-G. Ottaviani connecting failure of WLP to Laplace equations and classical results of Togliatti on osculating planes [17, 21]. We investigate WLP for quotients by quadratic monomial ideals, giving topological/algebraic reasons for failure of WLP in some cases not covered by [21, 22].

- Chris McDaniel, Larry Smith and Akihito Wachi met several times with Junzo Watanabe to further discuss his bold conjecture as well as an older paper of his which also seemed relevant to their work.
- Larry Smith met twice with Alexandra Seceleanu to discuss the methods we were using and their possible application to a problem of hers concerning symbolic powers of ideals and ideals of fat points.
- Larry Smith also met once with Hal Schenck to discuss some issues dealing with algebraic geometry that came up in their work.

5 Outcome of the Workshop

In our proposal for this workshop, we emphasized that the theory of the Lefschetz properties had many ties to other areas of mathematics, and the ties were sometimes amazing. At the end of the workshop we were even more convinced that this was indeed true.

It is truly remarkable that through the investigation of the Lefschetz properties of Artinian rings, some of the old long-forgotten papers are reviving. In the mid of 19th century P. Gordan and M. Noether [12] made an investigation on algebraic forms with vanishing Hessian, and it was followed by R. Permutti [25] and U. Perazzo [26, 27]. These investigations have not been in the main stream of algebraic geometry, but they have a long history and contain many subtle problems.

While the hypersurfaces with vanishing Hessian have a connection to Gorenstein rings which fail the strong Lefschetz property, the Laplace differential equations and Togliatti systems are related to Artinian algebras which fail the weak Lefschetz property.

As well as the revival of old papers, there is the emerging of new investigations in this area. As carefully explained in section 4, the participants of the workshop successfully introduced and made progress on new and promising projects, and as well made significant advances on existing projects. This was all done in an environment in which reports were made on a daily basis, with suggestions and comments from the other participants contributing to the endeavors. We briefly summarize.

The first group studied a generalization of the much used idealization construction to Artinian Gorenstein algebras with inverse system $\langle \sum x_i^e F_i \rangle$ with the F_i homogeneous of degree in a second set of variables. Using the bihomogeneity of these algebras they determined in many cases when these algebra satisfied WLP or SLP.

The second work group studied a geometric reproof by J. Vallés after L. Gruson of E. Togliatti's 1946 paper on Togliatti systems [35]. By understanding further the connection to cyclic Galois coverings of a surface they were able to extend his results from cubics to higher degrees d.

The third group found a surprising proof of the first problem they had posed: they showed that for any balanced 3-polytope P, the canonical Artinian reduction of the Stanley-Reisner ring $K[P]$ over an infinite field has the SLP in characteristic not 2 or 3. Their proof uses properties they established of balanced contractions of a simplicial 2-sphere and of colored polytopes.

The fourth group studied the Lefschetz properties of coinvariant rings. Chris McDaniel and Larry Smith announced their results on the embedding of complete intersections arising as coinvariant algebras of finite complex reflection groups into quadratic complete intersections, based on ideas coming from algebraic and differential topology originating in work of R. Bott and H. Samelson done in the 1950s [5]. This result is interesting, since based on work in invariant theory by R. Steinberg [34] it gives us a clue for how to describe a Macaulay dual generator of complete intersections. With the cooperation of Akahito Wachi they began at BIRS a project to extend their results that is ongoing.

It is a long-standing problem to characterize (or even to say anything about) the Macaulay dual generator of complete intersections. Some experts regarded this as an impossible problem, since the description becomes inevitably complicated. However, if two Artinian complete intersections with the same top degree are related by an inclusion, it gives us a natural question; how are their Macaulay dual generators related. It gives us a hope if we know one of them we know the other.

As explicitly indicated in the preceding pages, many new problems have been created in this workshop, and much progress made in their solutions, even while much work still remains to do. We have a strong desire to organize another such workshop in the near future to discuss these new problems.

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