DYNAMICS IN APPLIED FUNCTIONAL DIFFERENTIAL EQUATIONS

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1 Overview of the Field

The significance of Functional Differential Equations (FDEs), a large part of the broad field of Differential Equations, has grown enormously in the past two decades or so. They play a unique role and are of irreplaceable value in modelling real-world phenomena with aftereffects. The time delay effects are intrinsic features of numerous mechanisms of natural phenomena around us, and their mathematical models of functioning naturally lead to delay or functional differential equations; see, for example, relevant references in the monographs [2, 4, 8, 12]. FDEs are complex mathematical objects that provide in part a new insight into dynamical properties of differential equations. For example, simple FDEs can exhibit complicated dynamical behaviors which do not exist for corresponding ordinary differential equations. Many important theoretical issues for FDEs are well developed and described in classical monographs, such as [1, 5, 7]. Rapid and expansive growth of scholar activities in research and applications of FDEs in the past decade or so has led to new theoretical advances in the field as well as to their broad applicability in diverse areas. Particularly impressive has been the use of FDEs in applications. This is reflected in several recent monographs emphasizing applied aspects of FDEs together with relevant theoretical issues behind them [2, 4, 8, 12, 13].

2 Recent Developments and Open Problems

The principal goal of this project is to initiate and develop a long term collaborative research agenda between the four participants of the workshop. Such collaborative plans also imply a subsequent inclusion into the research program of graduate and postgraduate students at their home universities, as well as colleagues and collaborators working in adjacent areas of research in the field.

There is a range of important topics and open problems in the field, many of which are within the common interests of this group, and which the participants would like to consider for joint research as parts of this program. In very broad terms this project deals with the qualitative and numerical analyses of several classes of FDEs appearing in various problems of an applied nature. The research work will be extended to specific classes of differential delay equations, difference equations and discrete maps (in particular, to discretizations of continuous time FDEs), and FDEs subject to random perturbations. It will be aimed at studies of global dynamical properties, such as global asymptotic stability, periodicity, complex (chaotic) behaviors, and persistence of the dynamics under various perturbations.

The equations under consideration appear as mathematical models of numerous phenomena in physics, biology, economics, physiology, life sciences, and other fields. They also come from related problems of applied mathematics, such as wave processes described by hyperbolic partial differential equations. Extensive numerical simulations are expected to provide guidelines and directions for subsequent theoretical justification of observed dynamics. They will also serve as a means of verification between the theoretical results predicting the dynamics and available observed data for the corresponding real world phenomena.

The following general directions of research are included into collaborative plans between members of the team:

(i) Existence, stability, and shape of periodic solutions for functional differential equations and difference equations; associated non-autonomous dynamical systems, existence and properties of their global attractors;

(ii) Discretizations of FDEs, difference equations, discrete maps and their properties (global stability of equilibria, periodic solutions and their properties, chaotic behavior, general attractors);

(iii) Problems of optimization and optimal control for functional differential equations; Stabilization of dynamical equilibria, relevance to applications;

(iv) Deterministic and stochastic functional differential equations: global stability and existence of periodic solutions, persistence of dynamics under small stochastic perturbations, global dynamics;

(v) Applications to modeling real life phenomena in applied sciences (biology, physics, physiology), life sciences (economics, medicine, environmental problems), other areas of mathematics (e.g., boundary value problems for partial differential equations).

We have discussed those topics in general settings during the meeting and made an initial research progress on some of them.

3 Scientific Progress Made

One of the principal mathematical objects of the research exploration during the meeting was a cyclic system of delay differential equations of the form

$$\begin{array}{rcl} x_1'(t) + \lambda_1 x_1(t) & = & f_1(x_2(t-\tau_2)) \\ x_2'(t) + \lambda_2 x_2(t) & = & f_2(x_3(t-\tau_3)) \\ \dots & \dots & \dots \\ x_{n-1}'(t) + \lambda_{n-1} x_{n-1}(t) & = & f_{n-1}(x_n(t-\tau_n)) \\ x_n'(t) + \lambda_n x_n(t) & = & f_n(x_1(t-\tau_1)), \end{array} \tag{1}$$

where $\lambda_k \geq 0$ and $f_k \in C(\mathbf{R}, \mathbf{R})$. In addition, the nonlinearities f_k satisfy a sign condition $sign\,\{x \cdot f_k(x)\} = \sigma_k \in \{-1; +1\}$, allowing for system (1) to have zero as the only constant solution. In addition, we have assumed the eventual negative feedback in the system which can be expressed as $\sigma_1\sigma_2 \dots \sigma_n = -1$.

System (1) has numerous applications in modelling various real world phenomena. Just to mention a few, it was proposed as a mathematical model of a protein synthesis process where natural physiological delays are taken into account [3, 9]. Its two-dimensional version $n = 2$ was used as a model of intracellular circadian rhythm generator [10]. For other applications such as models of neural networks see e.g. [6, 13] and further references therein. Its one-dimensional case $n = 1$ is comprehensively studied in numerous publications, many of which are summarized as parts of several monographs, see e.g. [1, 2, 5, 8, 11, 12]. Those monographs also offer an extensive list of references to other applications. Though some theoretical results on system (1) are available (in particular, for the one-dimensional case $n = 1$), there are many theoretical and applied questions and open problems that still remain unsolved (especially for the higher dimensional case $n \geq 2$).

Our principal approach is to derive some of the basic properties of solutions of system (1) from those of the corresponding linear system

$$x_1'(t) + \lambda_1 x_1(t) \quad = \quad a_1\, x_2(t-\tau_2)$$

$$
\begin{aligned}
x_2'(t) + \lambda_2 x_2(t) &= a_2\, x_3(t-\tau_3) \\
\dots\quad \dots &\quad \dots \\
x_{n-1}'(t) + \lambda_{n-1} x_{n-1}(t) &= a_{n-1}\, x_n(t-\tau_n) \\
x_n'(t) + \lambda_n x_n(t) &= a_n\, x_1(t-\tau_1),
\end{aligned}
\tag{2}
$$

where $a_k := f_k'(0) \neq 0$. Many properties of solutions of system (2) are well studied and understood due to its linear nature. Those properties include the stability and instability of the zero solution, the oscillating nature of solutions, and the existence of monotone (non-oscillating) solutions. Associated with the linear system (2) is the so-called characteristic equation

$$
(z+\lambda_1)(z+\lambda_2)\dots(z+\lambda_n) = a\,\exp\{-\tau\, z\}, \quad a = a_1 a_2 \dots a_n. \tag{3}
$$

The location of zeros in the complex plane of the characteristic equation (3) completely determines multiple properties of the solutions of the linear system (2). We have shown that they also determine some of the properties for the nonlinear system (1). In particular we have shown the following:

- All solutions to system (1) oscillate if and only if the characteristic equation (3) does not have any real solutions;
- System (1) has a monotone (non-oscillating) solution if and only if the characteristic equation (3) has a real solution;
- There exist parameter values a_0 and a_1 such that all solutions to system (1) oscillate when $a > a_0$, and the zero solution is unstable when $a > a_1$. Both options $a_0 < a_1$ and $a_0 > a_1$ are possible in higher dimension $n \geq 3$ only.

One of the open problems that stands out in this direction of research for a number of years is the existence of the so-called slowly oscillating periodic solutions in a general setting. We conjecture that such periodic solutions exist for our system (1) when $a > \max\{a_0, a_1\}$ and at least one of the functions f_k is bounded from one side.

4 Outcome of the Meeting

A program of joint research is initiated and currently being further developed into a comprehensive research direction in this particular area of modern nonlinear dynamics. It includes qualitative analyses of autonomous systems of nonlinear delay differential equations modeling various real world phenomena. Besides the joint research efforts by the entire group as described above additional initial steps of further research exploration were undertaken between participants within the group on the following topics:

- Optimal control and optimization in a delay system modeling an economic production cycle (Ivanov and Trofimchuk);
- Existence and uniqueness of travelling wave solutions in partial differential equations models with delay (Hasik and Trofimchuk)
- Oscillation and stability of solutions in discrete systems with delay (Braverman and Ivanov)

We have made a substantial progress in our investigation of system (1) during the meeting itself. We have studied the corresponding characteristic equation (3) and related its properties to the solutions of the nonlinear system. Preparation of a typescript of a joint paper of the four co-authors is currently in progress. We expect to finish writing the typescript and to prepare it for submission by the end of this year.

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