The inverse eigenvalue problem of a graph

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1 Overview of the Field

Inverse eigenvalue problems appear in various contexts throughout mathematics and engineering, and refer to determining all possible lists of eigenvalues (spectra) for matrices fitting some description. The *inverse eigenvalue problem of a graph* refers to determining the possible spectra of real symmetric matrices whose pattern of nonzero off-diagonal entries is described by the edges of a given graph (precise definitions of this and other terms are given in the next paragraph). This problem and related variants have been of interest for many years and were originally approached through the study of ordered multiplicity lists. It was thought by many researchers in the field that at least for a tree T, determining the ordered multiplicity lists of T would suffice to determine the spectra of matrices described by T. When it was shown in [2] that this was not the case, the focus of much of the research in the area shifted to the narrower question of maximum multiplicity, or equivalently maximum nullity or minimum rank of matrices described by the graph. While the maximum multiplicity has been determined for many families of graphs, including all trees, in general it remains an open question and active area of research (see [4, 5] for extensive bibliographies). More recently, there has been progress on the related question of determining the minimum number of distinct eigenvalues of matrices described by a given graph [1, 3]. Maximum nullity, minimum number of distinct eigenvalues, and ordered multiplicity lists all provide information that can in some cases be used to solve the inverse eigenvalue problem for a specific graph, but the question of the structures or properties that are necessary to allow this to be done more generally is fundamental.

For a (simple, undirected) graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and edge set E, the *set of symmetric matrices described by* G, $\mathcal{S}(G)$, is the set of all real symmetric $n \times n$ matrices $A = [a_{ij}]$ such that for $i \neq j$, $a_{ij} \neq 0$ if and only if $ij \in E$. The maximum multiplicity of G is

$$M(G) = \max\{\mathrm{mult}_A(\lambda) : A \in \mathcal{S}(G),\ \lambda \in \mathrm{spec}(A)\}$$

and the minimum rank of G is $\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}$. It is easily seen that $M(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\}$ and $\mathrm{mr}(G) + M(G) = |G|$, where $|G|$ is the number of vertices of G. For a real symmetric matrix A, $q(A)$ denotes the number of distinct eigenvalues of A and for a graph G, the *minimum* *number of distinct eigenvalues of* G is $q(G) = \min\{q(A) : A \in \mathcal{S}(G)\}$. Given a real symmetric matrix A with distinct eigenvalues $\lambda_1 < \cdots < \lambda_r$ having multiplicity m_i for $\lambda_i, i = 1, \ldots, r$, the *ordered multiplicity list of* A is $(m_1, \ldots, m_r)$. For a graph G, the *set of ordered multiplicity lists of* G is the set of all ordered multiplicity lists of matrices $A \in \mathcal{S}(G)$. Observe that $q(G)$ is the minimum number of entries in an ordered multiplicity list of G and $M(G)$ is the maximum value of an entry in an ordered multiplicity list of G.

2 Recent Developments

At a research group meeting in Iowa in July 2015 attended by most of the participants of this FRG, several new tools were developed to attack the inverse eigenvalue problem of a graph [3]. These include the Strong Spectral Property (SSP) and Strong Multiplicity Property (SMP), matrix properties that generalize the Strong Arnold Property (SAP) (see [3] for precise definitions of the SSP and the SMP, and [6] for the SAP). All graphs having $q(G) \geq |G| - 1$ were characterized in [3].

3 Scientific Progress Made

While at BIRS we established several additional tools for the inverse eigenvalue problem of a graph: We defined and proved precise forms of minor monotonicity of SMP and SSP that preserve the multiplicities or eigenvalues of the minor in the larger graph but have some restrictions on additional multiplicities/eigenvalues. For matrices with SSP, we developed a method to produce another matrix with the same spectrum except one of the multiple eigenvalues has been split into one or more distinct eigenvalues.

We used minor monotonicity and subgraph monotonicity established in [3] to solve the inverse eigenvalue problem for graphs of order 5 (4 or less was known previously), and to determine the forbidden minors for graphs having at most one multiple eigenvalue; the latter are shown in Figure 1.

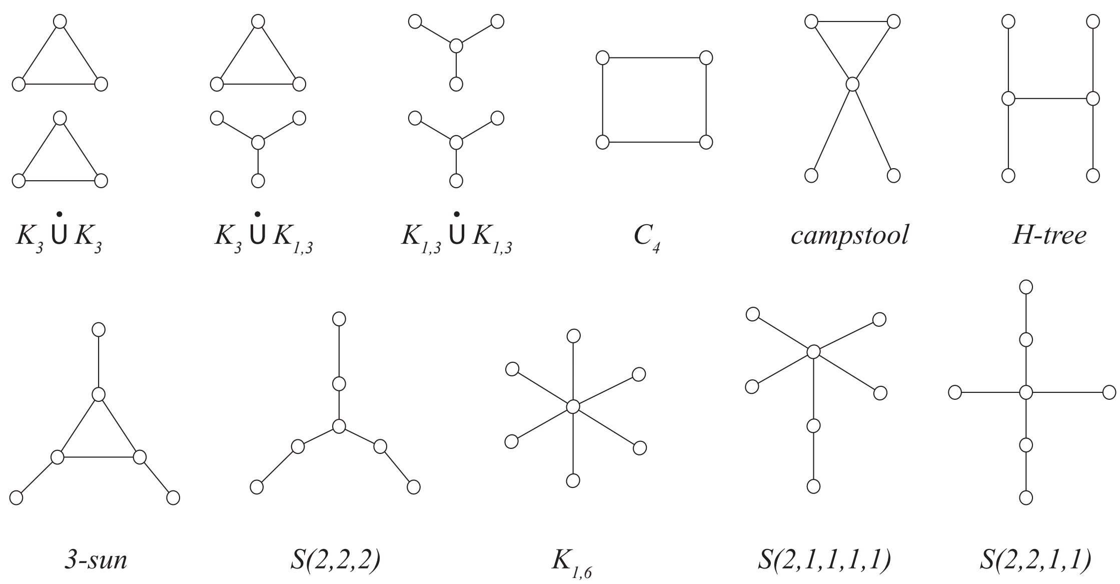

Figure 1: Any graph that has at least two multiple eigenvalues must have one of these 11 graphs as a minor.

4 Outcome of the Meeting

We are in the process of preparing a paper for submission to a journal describing the research discussed in Section 3; we expect to post a draft on arxiv by the end of 2016. We also made plans to work on extensions of SSP and SMP.

5 Acknowledgement

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