Topological Methods in Model Theory

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The event was quite productive, and the participants are grateful for BIRS for providing an excellent environment for their daily meetings during the week of June 12-19, 2016. The participants examined problems where topological methods can be used to improve or bring new insight into results in model theory. The following were among the projects considered.

1 Establishing a precise relationship between topological properties and model-theoretic properties.

The Baire Category Theorem and the Omitting Types Theorem for logics. One of the fundamental theorems of first-order model theory is the classical omitting types theorem. The proof of this which appears in textbooks uses the method known as "Henkin construction". However it is relatively well known that the theorem can be proved topologically, by using the Baire Category Theorem.

The topological approach to omitting types has the advantage that it can be adapted for contexts more general than first order. For instance, Morley [Mor74] used it to prove the Omitting Types Theorem for countable fragments of the infinitary logic $L_{\omega_1\omega}$, and Caicedo-Iovino [CI14] used it to provide a characterization of continuous logic. Later, Eagle [Eag14] used it to extend Morley's result to real-valued logics.

These developments have shown that the omitting types theorem holds in any abstract logic that has a space of structures which is Baire (i.e., it satisfies the Baire category theorem) and has a fairly simple syntax. It is then natural to ask for the most general situation under which conditions an abstract logic with a Baire space of structures satisfies the omitting types theorem, and whether the converse is true, i.e., whether every Baire space occurs as a space of structures of an abstract logic that satisfies the omitting types theorem.

During our week at BIRS, it became clear that merely assuming Baire is not sufficient [ET], and ongoing work by Eagle and Tall was sketched which associates logics to systems of topological spaces which aims to answer the second question.

Proposed project: To understand the logical contents of the game versions of Baire's theorem. As with the Banach-Mazur game, where Baire is equivalent to Player I not having a winning strategy while Player II having a winning strategy is equivalent to the stronger property of weak alpha-favourablity, one can similarly play games to omit types. The question of whether the omitting types theorem is equivalent to the prima facie stronger game version turns out to be strongly related to a long-unsolved problem in set theory, namely, the existence of P-filters on the set of natural numbers which are non-meagre subsets of the Cantor set. See [Mar98] and [MZ15].

Topological compactness and model-theoretic compactness. The expressive power of first-order logic is limited for some contexts, its model theory is powerful and mature. One of the characteristics of first-order model theory that make it particularly powerful is compactness. Proper extensions of first-order are typically not compact; however, several important extensions of first-order satisfy the weaker property of $[\kappa, \lambda]$-compactness, for infinite cardinals κ and λ.

The remarkable abstract compactness theorem, originally proved by Makowsky and Shelah in the 80's [MS83], states roughly that a logic $\mathscr{L}$ satisfies $[\kappa, \lambda]$-compactness if and only if there exists a (κ, λ)-regular ultrafilter D that provides an abstract notion of ultraproduct for structures of $\mathscr{L}$. The proof given by Makowsky and Shelah uses nontrivial model theory, and it supposes that the underlying logic $\mathscr{L}$ satisfies certain restrictive properties, e.g., $\mathscr{L}$ must be closed under negation.

At BIRS, we went through a simpler proof, due to Caicedo, that replaces the combinatorics in the Makowsky-Shelah analysis by elementary topological ideas related to preservation of topological $[\kappa, \lambda]$-compactness under products [Cai95]. This topological approach turns out to be not only simpler, but it also covers wider classes of logics, e.g., logics without classical negation.

Proposed project: Explore the relationship between this approach and current set-theoretic research on strong logics.

2 Topology and categoricity in model theory.

Morley's categoricity theorem [Mor65] is one of the groundbreaking theorems of model theory. It was the first instance of a *categoricity transfer theorem*. Roughly, a categoricity transfer result states that if a theory T is categorical in one cardinality (i.e. has only one model of that size), then T is categorical in other cardinalities. Research in this direction involves difficult combinatorial ideas [Bal09, She09b, She09a].

The motivation behind several of the methods originally used by Morley was topological; however, the spaces that he dealt with are 0-dimensional, so he was able to formulate them combinatorially for the benefit of non-topologically-minded model theorists.

We believe that approaching categoricity transfer results from a topological point of view, as Morley originally did, but using contemporary topological technology that was not available to him, will shed light onto this research direction in model theory. In particular, we noted an unexpected connection between the key model theoretic notion of stability and a topic extensively studied by topologists and set-theorists: superatomic Boolean algebras (see [Kop89, Section 17 of Chapter 6] and [MB89, Chapter 19]); the latter notion has a purely topological meaning when transferred to Boolean spaces of types, namely: the power of every quotient space is identical to its weight, which makes sense in non-Boolean contexts.

Proposed project: Explore the topic of superatomic Boolean algebras and their relation to categoricity.

3 Continuous operations between spaces of structures.

As shown by Caicedo [Cai95], the topology of elementary classes in spaces of structures associated to any Boolean model theoretic logic is uniformizable, and the uniform continuity of natural operations arising between them (Cartesian products, quotients, all sort of algebraic constructions, etc.) reflect the model theoretical properties of the logics. Thus, Lindström's axioms for logics, relativisations, interpretations, and Feferman-Vaught uniform reduction properties [FV59] are uniform continuity phenomena. The same is true of Beth's definability theorem and other definability properties.

Proposed project: Explore these connections in the realm of stronger logics.

4 Development of projective model theory and measurable logic.

The development of continuous logic in recent years has brought about fruitful interaction between model theory and other areas of mathematics, especially functional analysis. Continuous logic is finitary. However, the ideas of continuous model theory have been recently extended to the infinitary realm [BYI09, Eag14]. In finitary continuous logic, it is required that all the functions and relations of the structures involved be continuous, and that all the logical connectives be continuous functions as well. During the BIRS sessions, we realized that for an important property of the infinitary extension, namely, the omitting types property, the continuity of the structures is not needed (so the continuity of the connectives suffices). We intend to investigate the consequences of replacing the continuity of the logical connectives by a weaker condition without eliminating the (desirable) omitting types property. A natural candidate seems to be requiring that the connectives be projective in the sense of descriptive set theory [Kec95]. Formulas will then be closed under negation and the existential quantifier.

This project intersects with another: the development of "measurable logic". This has been a project in model theory for decades, since such a logic would be natural for applications to analysis. During our BIRS sessions, it was suggested that for the recent Dueñez-Iovino work on metastability [DI1, DI2], the continuous model theory formulation may be unnecessary, and projective logic would provide a finer foundation.

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