SYZ mirror symmetry

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1 Overview of the Field

Strominger-Yau-Zaslow (SYZ) [27] proposed that mirror symmetry can be understood in terms of duality of special Lagrangian torus fibrations. Namely, the mirror manifold can be constructed by taking fiberwise torus dual of the original manifold, and homological mirror symmetry can be understood by a real version of Fourier-Mukai transform. Fukaya [15] proposed that the mirror transform can be realized by family Lagrangian Floer theory of fibers of the SYZ fibration, which was further studied by Tu [28, 29] and Abouzaid [1, 2].

The SYZ program has led to a series of breakthroughs in geometric understanding of mirror symmetry. Based on Kontsevich-Soibelman's theory of wall-crossing [19], Gross-Siebert developed a general reconstruction program of mirror geometries using toric degenerations [18]. Auroux [4], Chan-Lau-Leung [8] and Abouzaid-Auroux-Katzarkov [3] realized the SYZ construction using symplecto-geometric methods and computed SYZ mirrors explicitly for local Calabi-Yau geometries. Moreover, Fukaya-Oh-Ohta-Ono [16] reconstructed the mirrors of toric manifolds based on the work of Cho-Oh [13] and proved close-string mirror symmetry for their geometrically constructed mirrors. Furthermore Aganagic-Vafa [5] proposed a generalized SYZ construction of mirror conifolds which uses non-compact Lagrangians constructed from knots in place of torus fibrations. Motivated from this and the works of Seidel [22, 23] and Sheridan [24, 25] on homological mirror symmetry, Cho-Hong-Lau [11, 12] developed a generalized SYZ program using Lagrangian immersions in place of torus fibrations.

2 Recent Developments and Open Problems

1. **Talk: Open Gromov-Witten invariants and augmentation varieties** (by Garrett Alston)
Talk: SYZ and Fourier-Mukai transform (by Naichung Leung)
Aganagic-Ekholm-Ng-Vafa [6] proposed a mathematical approach to prove that the generalized SYZ mirror of Aganagic-Vafa [5] is the augmentation variety associated to a knot. Briefly, the idea is the following. To a knot K in $\mathbb{S}^3$, one can associate a Legendrian submanifold in the unit sphere bundle of the cotangent bundle of $\mathbb{S}^3$. To this Legendrian is associated a differential graded algebra, which is a knot invariant. In turn, invariants associated to the dga, such as the augmentation variety, are also knot invariants. In another direction, the dga is related to non-compact Lagrangians in the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$, whose quantum-corrected moduli should give the mirror geometry. It was conjectured by [5] that the augmentation variety equals to the generalized SYZ mirror of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$. Alston and Leung in the group are working to prove the conjecture.

2. **Talk: SYZ and quantum deformations** (by Kwokwai Chan)
Chan-Leung-Ma [9] defined a differential graded Lie algebra in terms of Witten-Morse theory on a lattice bundle over the base of a Lagrangian torus fibration, which is isomorphic to the Kodaira-Spencer DGLA on the mirror Calabi-Yau manifold $\check{X}$ under the SYZ transform. It was speculated that the DGLA should control the deformation theory on the symplectic side, namely it should describe some kind of "quantum" deformations of the symplectic structure on X.

   Talk: Scattering diagrams and Witten-Morse theory (by Ziming Ma)
Moreover, the work [9] related scattering diagrams, which play a key role in Gross-Siebert program, with solutions to the B-model Maurer-Cartan equation using semiclassical expansions. However, the correspondence is away from the singular locus of the SYZ fibration. The ultimate goal is to extend it through singular locus to obtain a complete understanding of scattering in terms of Maurer-Cartan equation.

3. **Talk: SYZ and noncommutative deformations** (by Hansol Hong)
Kontsevich showed that there exists a canonical deformation quantization of the algebra of holomorphic functions on X with respect to a given Poisson structure α. On the other hand, one can transform α to be a 2-form on the symplectic manifold Y via SYZ transformation, which can be used to deform the symplectic structure on Y. The original torus fibers in Y may no longer be Lagrangian with respect to the deformed symplectic form. It is expected that one obtains a Lagrangian foliation after the deformation. Applying Connes theory, one obtains a noncommutative algebra from a foliation, which is expected to be mirror to the deformation quantization of X.

4. **Talk: Topological Fukaya category and localized mirror functor** (by Cheol-Hyun Cho)
Fukaya category of a symplectic manifold, which is a central object in homological mirror symmetry, is difficult to formulate. Recently, Kontsevich proposed that Fukaya category of a Stein manifold can be understood by gluing local data associated to those of a Lagrangian skeleton, which has the structure of cosheaf of dg categories. For punctured Riemann surfaces, Dyckerhoff-Kapranov [14], Sibilla-Truemann-Zaslow [26] and Pascaleff-Sibilla [21] constructed a notion called topological Fukaya categories and found applications in mirror symmetry. However, it is not yet proved that the topological Fukaya category is equivalent to the original Fukaya category.

5. **Talk: Generalized SYZ** (by Siu-Cheong Lau)
Cho-Hong-Lau [11, 12] developed a generalized SYZ program using Lagrangian immersions in place of Lagrangian torus fibrations. The program can be applied even in the absence of toric degenerations and Lagrangian fibrations, for instance rigid Calabi-Yau manifolds. Currently the program only concerns about formal deformations. Geometric deformations and family of immersions should be incorporated in the future. Moreover the program has more applications, for instance BHK transpose mirror symmetry [7, 20] and Hitchin systems [17].

3 Scientific Progress Made

1. Alston and Leung worked on an ongoing project to prove that the open Gromov-Witten potential function of a non-compact Lagrangian is related to the augmentation variety. They also discussed follow-up projects and applications to work on once this project is complete. The group discussed possible relations with the work [12] of Cho-Hong-Lau and non-commutative augmentations recently defined by [10]. There should exist a natural mirror functor from the Fukaya category of resolved conifold to the derived category of the moduli space of augmentations, which descends to the derived category of the augmentation variety.

2. Chan tried to get some idea on quantum deformations of symplectic geometry by studying J. Tu's recent works [28, 29, 30] on the reconstruction problem and the interpretation of homological mirror symmetry as a Koszul duality. The key idea is to make use of Fukaya's earlier work on constructing a sheaf of A_∞ algebras on B. We expect that this sheaf of A_∞ algebras is encoding the quantum information of the symplectic manifold (X, ω) because the symplectic structure ω appears as part of

the curvature term in the family of A_∞ structures. By studying the deformation theory of this sheaf of A_∞ algebras, perhaps also applying results analogue to that of Fukaya-Oh, one should be able to recover the DGLA we discussed above.

Moreover Ma found that it is better to consider the polyvector fields governing the extended deformation of holomorphic volume form. Barannikov gave a construction of B-model Frobenius structure on the extended complex moduli of a smooth Calabi-Yau manifold, from the universal solution to the extended Maurer-Cartan equation. For a one parameter family of K3 surfaces degenerating to the large complex structure limit, we use a resolution of the total space to set up a Barannikov type differential BV algebra, and explore the relation between tropical diagrams and the log-Frobenius manifold structure near the large complex structure limit.

3. Hong and Leung investigated the relation between noncommutative deformations of SYZ (Strominger-Yau-Zaslow) mirror pair, X and Y. More precisely, they considered the situation where X and Y admit Lagrangian torus fibations over the same base B, and the fibers in X and Y are dual to each other. They studied the detailed construction Lagrangian foliation structure on Y, and found that there is a way of expanding the foliation in Fourier series in a suitable sense. The observation is based on the fact that SYZ transformation is, fiberwisely, a Fourier transform. While the genuine foliation could be too complicated to handle, one may still be able to find a construction of an noncommutative algebra (in the sense of Connes) directly from the Fourier expansion data.

4. Cho, Hong and Lau discussed the relationship between topological Fukaya category and localized mirror functors in [11]. Although the topological Fukaya category is made from perfect complexes of A-type quiver representations, we have learned that localized mirror functor formalism in the Calabi-Yau setting should be very helpful to find the isomorphism between the original Fukaya category and the topological one, after one establishes the definition of appropriate Fukaya category of a Riemann surface with non-empty boundaries together with marked points.

5. They also discussed a way to understand BHK transpose mirror symmetry using the general theory of [11]. Given a CY hypersurface defined by an invertible polynomial, they constructed Lagrangian immersions in the hypersurface and discussed how transpose polynomial arises by counting certain orbi-polygons bounded by the Lagrangian immersions. It was found that deformations by twisted sectors are necessary in order to have the transpose terms. This may give a hint on understanding transpose mirror symmetry for general-type hypersurfaces as well.

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