# GROUPS AND GEOMETRIES

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September 2-7, 2012

## 1 Overview of the field, and recent developments

As groups are just the mathematical way to investigate symmetries, it is clear that a significant number of problems from various areas of mathematics can be translated into specialized problems about finite permutation groups, linear groups, algebraic groups, and so on. In order to go about solving these problems a good understanding of the finite groups, especially the simple ones, is necessary. Examples of this procedure can be found in questions arising from algebraic geometry, in applications to the study of algebraic curves, in communication theory, in arithmetic groups, model theory, computational algebra and random walks in Markov theory. Hence it is important to improve our understanding of groups in order to be able to answer the questions raised by all these areas of application.

The research areas covered at the meeting falls into three main areas.

### 1.1 Fusion systems and new approaches to the classification of finite simple groups

A fusion system over a p-group S is a category whose objects form the set of all subgroups of S, whose morphisms are certain injective group homomorphisms, and which satisfies axioms that are modelled on conjugacy relations in finite groups. The definition was originally motivated by representation theory, but fusion systems also have applications to local group theory and to homotopy theory. It is on the applications to finite group theory that we focus here. In this respect the basic theory – such as how to define normal subsystems of fusion systems, and so on – has only been developed in the past few years (see for example[1], [10]). Yet there are now profound, if currently speculative, ideas of Aschbacher, Chermak and others about a possible program to use the theory of fusion systems to simplify a huge part of the classification of finite simple groups which deals with the groups of component type.

Another new approach is the Meierfrankenfeld-Stellmacher-Stroth program [11]. For a prime $p$, one says that a finite group $G$ has characteristic $p$ if $C_G(O_p(G)) \leq O_p(G)$, where $O_p(G)$ is the largest normal $p$-subgroup of G; and $G$ is said to have local characteristic $p$ if every $p$-local subgroup of $G$ has characteristic $p$. Broadly speaking, the MSS program attempts to understand (and classify) the finite groups of local characteristic $p$. Success in this program would replace a huge part of the original classification of finite simple groups. The methods involved are diverse, involving amalgams, representation theory as well as local group theory.

Finally, there is of course the original Gorenstein-Lyons-Solomon program [5], which aims at giving a more or less self contained proof of the classification of finite simple groups within eleven monographs. The first six have been completed and published, and much of the material for the remaining volumes is currently in manuscript form. Two of the outstanding problems remaining to be solved are the well-known "$e(G) = 3$" and "uniqueness"cases, on which several of the conference participants are working.

## 1.2 Geometry and groups

There are many ways in which group theory is related to geometry. The most appropriate geometric tools for the investigation of finite simple groups are their actions on combinatorial objects, in particular buildings. The recent highlights of this interaction are the notion of complete reducibility introduced by Serre in around 2000 which yielded a proof of the center conjecture in 2009, and the investigation of Phan-presentations of finite simple groups, which constituted the starting point for the proof of the rank conjecture for arithmetic groups over function fields.

The importance of buildings for finite group theory is due to the fact that they provide an efficient tool to study exceptional groups of Lie-type. Most beautiful examples for this are the recent application of Tits' extension theorem for spherical buildings in the MSS-programme and the theory of Phan-presentations for exceptional groups. In this context, Van Maldeghem and his collaborators are currently working on a program which is designed to provide more combinatorial information about exceptional groups by a suitable interpretation of the Freudenthal-Tits magic square in the split case. Another focus of current research is affine buildings and in particular trees. Using geometric methods Struyve was able to accomplish the solution of the existence problem for affine buildings which was open since the development of Bruhat-Tits theory in the 1970. Finite group theory is most important in the investigation of locally finite affine buildings. Recent examples are provided by the work of Caprace and De Medts on Moufang sets at infinity of a tree and Grüninger's work on Moufang sets at infinity of affine Moufang twin trees.

## 1.3 Applications of simple group theory

There are many applications of the theory of simple groups, particularly the classification theorem of finite simple groups. Here we discuss two such recent applications.

The first application concerns the theory of expansion in finite groups. A famous result of Helfgott [7] asserts that arbitrary generating sets $A$ in $SL(2, p)$ expand, in the sense that the product of three copies of $A$ is either significantly larger than $A$, or is already equal to the whole group. This result has turned out to have quite a range of applications, from the theory of expander graphs to the development of new sieve methods in analytic number theory. Recently, Breuillard, Green and Tao [3], and independently Pyber and Szabo [12], have spectacularly generalised Helfgott's result to all families of finite groups of Lie type, using methods of additive combinatorics and model theory as well as the structure theory of the groups. These methods seem to offer scope for further applications, both within and outside group theory.

The second application concerns algebraic geometry. Guralnick and Tiep [6] have recently proved a number of results, such as the Kollar-Larsen conjecture on holonomy groups of vector bundles on a smooth projective variety. This was done via work of Kollar, Larsen and Katz which translated the conjecture to a question about subgroups of $GL(V)$ which are irreducible on some symmetric power of the vector space $V$, which was solved by Guralnick and Tiep using detailed structure theory of simple groups together with representation theory and some theory of algebraic groups. They have also proved results on crepant resolutions and quotients of Calabi-Yau varieties, again via translations to group theoretic problems. It seems that there is scope for further applications of group theory to algebraic geometry in this direction.

# 2 Presentation Highlights

## 2.1 Fusion systems and the classification

One of the main highlights of the whole meeting was Michael Aschbacher's talk *Fusion systems and groups of component type*. In the talk, Aschbacher discussed a speculative program to use the theory of fusion systems to simplify that part of the classification of the finite simple groups dealing with the groups of component type. Many participants expressed interest in developing the ideas for this program.

Complementing Aschbacher's talk was the lecture of Andy Chermak, *The normal structure of linking systems*. He discussed the category of partial groups and the subcategory of objective partial groups. Among these are the finite group-like "centric linking systems". He focussed on normal subsystems of linking systems and on the problem of how to view these as linking systems in their own right. The results outlined are in analogy with Aschbacher's theory of normal subsystems of fusion systems.

Another highlight in this area was the lecture of Ellen Henke, *Cohomology, $F$-Isomorphism and Fusion in Finite Groups*. This talk was about the proof of a general group theoretical result which has implications for mod $p$ cohomology and higher chromatic cohomology theories. The result is as follows: suppose we are given a finite group $G$, an odd prime $p$, and a subgroup $H$ of $G$ containing a Sylow $p$-subgroup of $G$. Then $H$ controls fusion in $G$ if and only if it controls fusion of elementary abelian subgroups. The analogous result is true for $p = 2$ if one considers abelian subgroups of exponent at most $4$ instead of elementary abelian subgroups. The most striking fact about the proof is is that it is carried out in the category of fusion systems rather than in the category of groups.

Concerning other parts of the revision program, there were talks by Ron Solomon, Kay Magaard and Chris Parker. Magaard ( *Groups of even type which are not of even characteristic*) and Parker ( *Groups which are almost Lie type*) talked about recent contributions to the Meirerfrankenfeld-Stellmacher-Stroth program, while Solomon (*Characterizing Lie Type Groups*) reported on a recent major theorem with Richard Lyons which forms an important part of the Gorenstein-Lyons-Solomon program.

## 2.2 Geometry and groups

One of the main themes in this area discussed at the meeting was the topic of buildings. A highlight was the talk of Richard Weiss (*The local structure of Bruhat-Tits buildings*). Roughly speaking, Bruhat-Tits buildings are classified by spherical buildings defined over a field $K$ that is complete with respect to a discrete valuation. In particular, each spherical building $\Delta$ defined over a complete field $K$ is the "building at infinity" of a unique Bruhat-Tits building $X$, and the residues of the building $X$ are spherical buildings defined over the residue field $\bar{K}$. Bruhat-Tits buildings are not uniquely determined by their residues (in contrast to spherical buildings), but their residues are nevertheless an important structural feature of these buildings. Weiss discussed efforts to give a complete description of all the possibilities that arise for each family of spherical buildings $\Delta$ defined over a complete field $K$.

Another talk in this area was that of Koen Struyve, *Galois descent of Bruhat-Tits buildings*. Bruhat-Tits buildings are a class of affine buildings associated to certain classical, algebraic and mixed groups defined over (skew) fields with a complete valuation. An open question in Bruhat-Tits theory concerned the existence of Bruhat-Tits buildings for these groups, with the remaining open cases pertaining to certain exceptional groups of relative rank 1 and 2. In his talk Struyve announced a general solution of this problem using combinatorial and geometric methods.

Hendrik van Maldeghem, in his talk *Characterizations of groups by geometries and geometries by groups*, presented two recent characterizations of groups and geometries related to buildings. The first one characterizes the "standard orbits" of certain modular representations of the groups of the second row of the Freudenthal-Tits Magic Square by a simple extension of the Mazzocca-Melone axioms that were designed for quadric Veronesean varieties (corresponding, however, to the first cell of the second row of the FT Magic Square). The second characterization is one of the finite Hermitian unital in the spirit of the "Moufang property" for projective planes (and, more generally, twin buildings).

Another highlight on a geometric theme was the talk of Pierre-Emanuel Caprace, *Simple locally compact* *groups and branching*. The global structure of a Lie group is largely determined by its local structure, encoded in the Lie algebra. In the case of locally compact groups that are non-discrete and totally disconnected, the local structure is given by a commensurability class of profinite groups. Caprace's results illustrate that, when the ambient group is simple and compactly generated, there is also a local-to-global correspondence in that case, although it is not as tight as in Lie theory. This gives rise to the tantalising prospect of a possible classification program for certain types of simple, locally compact, totally disconnected groups.

Other geometric talks were given by Arjeh Cohen (*Constructing Riemann surface models from regular maps*) and Jonathan Hall (*Algebras from groups and geometry*). Cohen described computational methods for finding algebraic models of the smooth complex projective curves (Riemann surfaces) connected to regular maps, and Hall discussed recent progress on some elegant connections between the concept of triality in Lie theory and various types of alternative algebras, tracing the topic from pioneering work of Study, Cartan and Moufang to the present.

## 2.3 Applications

One of the highlights here was Laci Pyber's talk *Growth in linear groups* on his recent spectacular results on growth. His well-known "Product theorem" with Szabo [12] (proved independently by Breuillard, Green and Tao [3]) showed that for a generating set S of a finite group of Lie type $G$ of bounded rank, either $S$ "grows" (in the sense that $|S^3| > |S|^{1+\epsilon}$ for some absolute constant $\epsilon > 0$), or $S^3$ is the whole of $G$. This result has had a huge impact, for example proving Babai's conjecture on the poly-logarithmic diameter of Cayley graphs for such groups $G$, and leading to new families of expander graphs. Pyber's new results (also with Szabo) give precise structural information about arbitrrary (not necessarily generating) subsets $S$ which do not grow. Let $S$ be an inverse-closed subset of $\mathrm{GL}(n, F)$ satisfying $|S^3| < K|S|$ for some $K > 1$ , where $F$ is an arbitrary field. Then $S$ is contained in the union of polynomially many (more precisely $K^{c(n)}$ ) cosets of a finite-by-soluble subgroup $G$ normalised by $S$. Moreover $G$ has a finite subgroup $P$ normalised by $S$ such that $G/P$ is soluble and $S^3$ contains a coset of $P$. This includes the Product theorem mentioned above.

Another talk in this area was given by Nick Gill, *The width of a finite simple group*. He described recent work with Pyber, Short and Szabo on the "Product Decomposition Conjecture" of Liebeck, Nikolov and Shalev [9]. Given a finite simple group $G$ and a subset $A$ of $G$, the conjecture states that $G$ can be written as a product of the order of $\log |G|/ \log |A|$ conjugates of $A$. Gill discussed progress on this conjecture, using the Product Theorem of Pyber's talk, and uncovered some interesting new connections between classical additive combinatorics and normal subsets of a group.

Another highlight was the talk of Pham Huu Tiep on his joint work with Guralnick [6], which uses the theory of finite and algebraic linear groups to study questions in algebraic geometry. In particular they have proved some conjectures of Kollar and Larsen concerning stability of vector bundles, quotients of Calabi-Yau varieties, and (non-)existence of crepant resolutions.

On another front, Robert Guralnick (*Maximal Subgroups of Finite Groups*) talked about some extraordinary and unexpected new results on first cohomology groups of simple groups with coefficients in an irreducible module. In many cases the dimensions of these are given by coefficients of Kazhdan-Lusztig polynomials, which are notoriously difficult to compute, but there has been a recent breakthrough. This links up with questions about counting the number of maximal subgroups and the number of conjugacy classes of maximal subgroups of finite groups. The problem naturally splits up into first looking at maximal subgroups of almost simple groups and secondly getting certain bounds on the size of first cohomology groups. Guralnick discussed results with Larsen and Tiep on both problems. In particular, the recent cohomology results imply, most unexpectedly, that Wall's conjecture – stating that the number of maximal subgroups of a finite group $G$ is less than $|G|$ – is false.

There were also several talks on applications to permutation groups by Cheryl Praeger (*Coprime subdegrees for primitive permutation groups and completely reducible linear groups*), Tim Burness (*Bases for algebraic groups*) and Barbara Baumeister (*Permutation Groups and Applications*). Praeger duscussed a remarkable joint result with Dolfi, Guralnick and Spiga [4]: if a group $H$ acts irreducibly on a finite vector space $V$, then for every pair of non-zero vectors, their orbit lengths have a nontrivial common factor. Burness talked about bases of permutation groups: if $G$ is a permutation group on a set $X$, a subset $B$ of $X$ is a base for $G$ if the pointwise stabilizer of $B$ in $G$ is trivial. The minimal size of a base is called the base size of $G$. Bases for finite permutation groups have been widely studied since the nineteenth century, with many new results and applications in recent years. Burness discussed his recent work with with Guralnick and Saxl on bases for primitive actions of simple algebraic groups over algebraically closed fields, computing the exact base size in almost all cases. This leads to new results on base sizes for the corresponding finite groups of Lie type. Baumeister presented examples, applications and characterizations of permutation polytopes.

# 3 Scientific Progress Made and Outcome of the Meeting

The presentations given at the meeting and the lively discussions there revealed that there are various important directions of research which are highly interrelated. The exciting developments in the area of fusion system provide new stimulation and insights for the ongoing programs which are closely related to the revision of the classification of the finite simple groups. In particular the talk of Aschbacher elucidated that there is a need to bring both fields closer together. The participants of the meeting were convinced that there is a high potential to make substantial progress in both areas by joining forces.

Concerning the Meierfrankenfeld-Stellmacher-Stroth program, the talk of Parker proved that the endgame – identifying the groups in question – is in good shape, and so we can say there is a promising roadmap for the program. Nevertheless there are still large parts remaining to be done. As Stellmacher told us during the meeting, there is work in progress on these missing parts. Moreover Magaard reported in his talk about new results, which might replace the assumption of local characteristic $p$ with parabolic characteristic $p$. This would be important in both projects, the MSS and the Gorenstein-Lyons-Solomon programs. It will provide us with a bridge between them. To use this there must be some alterations in the MSS-program, and there is work in progress on this. Ron Solomon presented recent progress concerning Gorenstein-Lyons-Solomon program. The highlight of his talk was a theorem that essentially gives a characterization of large alternating groups and groups of Lie type of higher ranks. This result will be the final milestone for the completion of volume 7 of the GLS-series. His talk was followed by further discussions by the participants involved in the GLS-program about the progress of the program and futher steps to be taken.

Caprace presented in his talk an exciting new perspective by highlighting the importance of the classification of finite simple groups for the theory of totally disconnected locally compact groups. Groups acting on locally finite buildings provide many important examples of these. The participants working on geometric aspects were especially enthusiastic about this new direction of research. There is a high chance that the material in Caprace's talk marks the starting point for a new program on locally finite buildings, since it reveals questions and problems that had not yet been considered at all. Although the expectations are very high there is still a lot be explored in order to estimate the true impact of this development.

Also concerning the applications of simple group theory, we mention in particular the talk of Tiep in which he described some quite unexpected applications of linear groups to conjectures in algebraic geometry. Following on from this, there is scope for extending these developments into a whole program working towards much more general conjectures, and several participants expressed interest in joining such a program.

A geometric program which is also still in its early stages was presented in Van Maldeghem's talk on the Freudenthal-Tits magic square. Here there are already concrete contributions on the geometry of the related buildings. The remarkable application of giving a higher rank analogue of the Mazocca-Melone characterization for the quadratic Veronesian varieties provides also new geometric insights into this classical work. Several participants expressed their opinion that further work on this ambitious programme will yield much stronger results. There are indeed several problems about geometries related to exceptional groups which have been open more than 20 years – for example questions about embeddability of certain point-line spaces and existence of certain ovoids. Solutions to some of these questions seem now to be within reach. The participants working on Moufang sets expressed their confidence that this work will give a geometric – and thus more concrete – approach to Moufang sets related to exceptional groups. The absence of such an approach is a major obstacle in their attempts to get further in their efforts towards a classification.

Another major geometric theme was the structure theory of affine buildings. Here the emphasis was mainly on the applications of geometric tools which were developed earlier in the context of simple groups of Lie type. Weiss presented in his talk a local structure theory for Bruhat-Tits buildings of type $\tilde{B}_2$. For

Bruhat-Tits buildings of rank at least 3, this is by far the hardest case, since there are particular phenomena which cause problems in the context of exceptional quadrangles. One important insight provided by the solution of these problems is that certain exotic examples of Lie-type groups have a natural meaning in the context of semi-simple algebraic groups. During the conference the question about the impact on the theory of semi-simple groups came up as a direction to be explored further.

# 4 Final remarks

One major goal of the conference was to bring together researchers who are working in different directions which are closely related to finite simple groups. Apart from maintaining the scientific exchange between these areas there was the intention of creating new perspectives in research and stimulating scientific collaboration. The lectures and the time schedule were designed by the organizers in accordance with these objectives. In the selection of the lectures preference was given to subjects which offered the participants the possibility to learn about new developments in an area. It was also made sure that there remained plenty of time for discussion between the talks.

The feedback of many participants to the organizers was very positive. The outstanding quality of the talks was often mentioned, as well as many new collaborations started and ongoing ones continued. Beside the outstanding scientific level, people enjoyed following the lectures because the speakers succeeded in presenting topics by explaining clearly the main ideas and avoiding technical details. It was also remarked positively, that there was a comparatively high number of young speakers and that all of them gave beautiful lectures.

Finally, we remark that of the 42 participants, 11 were female.

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