“Manifolds with special holonomy and their calibrated submanifolds and connections”

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1 Overview of the field

The *holonomy* H of an oriented Riemannian manifold (M, g) of dimension n is a compact Lie subgroup of $\mathrm{SO}(n)$, which is a global invariant that is intimately related to the Riemann curvature tensor of g, via the Ambrose-Singer theorem. More precisely, its Lie algebra $\mathfrak{h}$ is generated by the Riemann curvature tensor R of the metric. Because of this, metrics with reduced holonomy (a proper subgroup of $\mathrm{SO}(n)$) have restrictions on their curvature, which makes them interesting solutions to certain prescribed curvature equations. Note that the holonomy condition is actually a *first order* condition on the metric, which automatically implies a second order condition. In 1955, Marcel Berger classified the possible Riemannian holonomy groups that can occur. In the case that M is not locally reducible and not locally symmetric, he found that only seven possible holonomy groups could occur. These groups are summarized in Table 1.

In particular, the last two examples are called the *exceptional* holonomy groups, as they occur in particular dimensions and are related to exceptional structures in algebra (the octonions.) It was initially thought that, although Berger could not exclude these possibilities, they would not actually occur. This was proved to not be the case, as Bryant found the first local examples in the 1980’s, followed by complete non-compact examples by Bryant-Salamon and independently by groups of physicists, and later compact examples by Joyce in 1994. The last five holonomies in Table 1 (all but the generic and Kähler holonomies) are often called *special* holonomies. They are also characterized

Holonomy group	n	Name	Remarks
$\mathrm{SO}(n)$	n	oriented Riemannian	generic
$\mathrm{U}(m)$	$2m$	Kähler	complex and symplectic
$\mathrm{SU}(m)$	$2m$	Calabi-Yau	Ricci flat and Kähler
$\mathrm{Sp}(m)$	$4m$	hyperKähler	Calabi-Yau (Ricci flat) in an S^2 family of ways
$\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$	$4m$	quaternionic-Kähler	positive Einstein (but *not* Kähler)
G_2	7	G_2 manifolds	Ricci-flat, related to octonion algebra
$\mathrm{Spin}(7)$	8	$\mathrm{Spin}(7)$ manifolds	Ricci-flat, related to octonion algebra

Table 1: The possible Riemannian holonomy groups

by the fact that they admit parallel or Killing spinors, which are important ingredients in theories of physics that incorporate supersymmetry. As a result, such metrics have long been of intense interest in physics. All metrics with reduced holonomy come equipped with one or more differential forms which are parallel with respect to the Levi-Civita connection. These forms are intimately related to the concept of a *calibration*, which we discuss next.

A *calibration* α on a Riemannian manifold (M, g) is a closed k-form satisfying a certain inequality related to the metric g, called the *comass one condition*. Given a calibration, we say that an oriented k-dimensional submanifold L of M is a *calibrated submanifold* if the form α restricts on L to the induced volume form. A theorem of Harvey–Lawson says that such submanifolds are always minimal (that is, they have vanishing mean curvature). Again we see an example where a first order condition automatically implies a second order condition. In addition, the known examples of calibrations for which there exist many non-trivial calibrated submanifolds are defined on manifolds with special holonomy. The most studied calibrations are summarized in Table 2.

Ambient manifold	k	Calibration form	Name of calibrated submanifolds
Kähler	$2p$	$\frac{1}{p!}\omega^p$	Kähler submanifolds
Calabi-Yau	$m = \frac{n}{2}$	$\mathrm{Re}(e^{i\theta}\Omega)$	special Lagrangian submanifolds of phase $e^{i\theta}$
G_2	3	φ	associative submanifolds
G_2	4	ψ	coassociative submanifolds
$\mathrm{Spin}(7)$	4	Φ	Cayley submanifolds
quaternionic-Kähler	4p	Υ^p	quaternionic-Kähler submanifolds

Table 2: The known calibrations which admits many calibrated submanifolds

Many explicit examples of calibrated submanifolds are known, both in Euclidean spaces and in complete non-compact manifolds. These usually involve a high degree of symmetry, often reducing the non-linear system of partial differential equations to an integrable ordinary differential equation. There are also examples of Kähler and special Lagrangian submanifolds in compact ambient manifolds. A very small subset of the mathematicians who have worked in this area include Bryant, Joyce, Harvey, and Lawson. It is interesting to note that calibrated submanifolds seem to belong to two different families, exhibiting strikingly different behaviour. For example, work of McLean has shown that the moduli spaces of special Lagrangian and of coassociative submanifolds are smooth and unobstructed, and intrinsically defined. This is definitely not the case for Kähler, associative, and Cayley submanifolds. Properties of the moduli spaces of special Lagrangian submanifolds were also extensively studied by Hitchin. The two families of calibrated submanifolds are called *instantons* and *branes* by Leung and his collaborators, who derive many common properties of such submanifolds. Calibrated submanifolds also play a crucial role in the phenomenon known as *mirror symmetry*, which we discuss further below.

Finally, on manifolds with special holonomy, one can consider (real or complex) vector bundles over them, and in some cases, the parallel (calibrating) differential forms allows one to define a special class of connections. Such *calibrated connections* are essentially defined by the fact that the curvature 2-form F_A of the connection A lies in the Lie algebra $\mathfrak{h}$ of the holonomy group H, which is a natural subspace of the space of 2-forms. Examples include Hermitian-Yang-Mills or Hermitian-Einstein connections on bundles over Kähler manifolds, and the Donaldson-Thomas connections on G_2 and $\mathrm{Spin}(7)$ manifolds. These connections can all be viewed as higher-dimensional analogues of the anti-self-dual connections (ASD) over Riemannian 4-manifolds, which were the key ingredients in the spectacular success of the work of Donaldson, Taubes, Uhlenbeck, and others in the early 1980's. Calibrated connections in the G_2 and $\mathrm{Spin}(7)$ settings are still not as well understood as they are in the Kähler setting.

Manifolds with special holonomy are believed to exhibit the phenomenon of mirror symmetry, which is currently best understood in the hyperKähler and Calabi-Yau cases, but which for the exceptional cases is at present more mysterious. Due to work of Strominger–Yau–Zaslow, Gukov–Yau–Zaslow, and others, it is expected that understanding the geometric aspects of mirror symmetry will involve studying the moduli spaces of calibrated submanifolds that these manifolds possess, as well as the moduli spaces of ASD connections or their exceptional holonomy analogues. A bit more precisely, it is expected that under certain conditions, a manifold M with special holonomy will fibre over a base space B, with the generic fibre being a calibrated torus in M. We know that such a fibration *must* have singular fibres, which are very difficult to deal with analytically and geometrically. The idea is that the "mirror" $\widehat{M}$ should be obtainable by "dualizing" the smooth part of this fibration, and then somehow compactifying to deal with the singular fibres. Although a great deal of progress has been made in the case of mirror symmetry for Calabi-Yau manifolds, by Gross, Ruan, Seidel, and others, we are still far from a complete proof of the "Strominger-Yau-Zaslow conjecture" (which is not even precisely formulated.) There is much less progress on mirror symmetry for the exceptional holonomy groups, other than some important early papers by Acharya and other physicists.

Another important aspect is the role of spinors and Dirac operators in these settings. Spin geometry seems to be natural for describing many of these structures. For example, we have already mentioned that the Ricci-flat manifolds that have special holonomy admit parallel spinors. Work of Harvey and others shows that calibrations can be obtained as the "square" of a spinor. It is likely that spinors may also prove to be fruitful in the study of calibrated connections.

2 Objectives of the workshop

It is important to note that mirror symmetry was (successfully) predicted by physicists, working in string theory and M-theory. The intuition of the physics community has been absolutely essential for so much of the spectacular progress that has been made in differential geometry and algebraic geometry since the 1970's, led by Atiyah, Bott, Witten, Yau, and others, and this continues to be the case. We name just a few of the research areas in geometry that are heavily influenced and inspired by physics: Lorentzian geometry (Positive Mass Theorem, Penrose Inequality), connections on vector bundles (Yang-Mills theory, Donaldson theory, Seiberg-Witten theory), enumerative algebraic geometry (Gromov-Witten invariants, quantum cohomology), Einstein metrics (the AdS / CFT correspondence, stability of vector bundles), and many others.

It is therefore vitally important for the geometry community to continue to maintain and to expand their dialogue with the physics community, for the mutual benefit of both parties. This particular BIRS workshop served just such a purpose, gathering together geometers and physicists, and allowing them to work together in an environment that was ideally suited for making important research advances. We expect that there will be several new collaborations arising from this workshop. It was also very important to us that we invited a significant number of young researchers, both graduate students and postdoctoral fellows, to expose them to some of the current research in this area, and to enable them to meet other mathematicians and begin new research projects. No less than 8 out of the 35 participants were either graduate students or postdoctoral fellows.

3 Summary of the talks

There were a number of talks given at the workshop. Here are the abstracts, in alphabetical order by speaker surname:

Speaker: **Bielawski, Roger (University of Leeds)**
Title: **Pluricomplex geometry and quaternionic manifolds**
Abstract: I will describe a new type of geometric structure on complex manifolds. It can be viewed as a deformation of a hypercomplex structure, but it also leads to special types of hypercomplex and hyper-Kähler geometry. These structures have both algebro-geometric and differential-geometric descriptions, and there are interesting examples arising from physics. Moreover, a class of pluricomplex manifolds leads to quaternionic-Kähler metrics, generalising the $\mathrm{SO}(3)$-invariant self-dual

Einstein examples of Hitchin.

Speaker: **Chen, Yunxia (Chinese University of Hong Kong)**
Title: **Minuscule representation bundles on surfaces with ADE singularities**
Abstract: The minimal resolution of a surface with a simple singularity has a bunch of (-2)-curves as its exceptional locus, whose dual graph is a Dynkin diagram of type ADE. In this talk, we construct minuscule representations of the corresponding Lie algebra using configurations of (-1)-curves. Then we build extension bundles over the resolution using their associated line bundles satisfying: (i) they can be descended to the singular surface and (ii) they carry natural tensorial structures. Using (ii) we construct ADE Lie algebra bundles so that the original vector bundles become minuscule ADE representation bundles over our surface with ADE singularities.

Speaker: **Cherkis, Sergey (University of Arizona)**
Title: **Octonions, monopoles, and knots**
Abstract: Witten's approach to Khovanov homology of knots is based on the five-dimensional system of equations, which we call the Haydys–Witten equations. We formulate a dual seven-dimensional system of equations. It can be formulated on any G_2 holonomy manifold and is a close cousin of the monopole equation of Bogomolny. The octonions play a central role in our view of the Haydys–Witten equations and in the transform relating the five- and seven-dinemsional systems.

Speaker: **Conlon, Ronan (McMaster University)**
Title: **A theorem of existence for asymptotically conical Calabi-Yau manifolds**
Abstract: Asymptotically conical (AC) Calabi-Yau manifolds are Ricci-flat Kähler manifolds that resemble a Ricci-flat Kähler cone at infinity. I will describe an existence theorem for AC Calabi-Yau manifolds which, in particular, yields a refinement of an existence theorem of Tian and Yau for such manifolds. I will also discuss some examples. This is ongoing work with Hans-Joachim Hein.

Speaker: **Dunajski, Maciej (University of Cambridge)**
Title: G_2 **geometry, twistor theory, and cuspidal cubics**
Abstract: We establish a twistor correspondence between seven-parameter families of rational curves in a surface, and certain G_2 structures on moduli spaces of such curves. There are several explicit examples — e.g. the space of all cuspidal cubic curves in $\mathbb{P}^2$ gives rise to a homogenous co-calibrated G_2 structure on $\mathrm{SU}(2,1)/\mathrm{U}(1)$.

Speaker: **Gayet, Damien (Université Lyon I)**
Title: **Smoothing moduli spaces of associative submanifolds**
Abstract: It is known that deforming a closed associative Y (respectively an associative Y with boundary in a fixed coassociative X) as an associative is an elliptic problem of vanishing index (respectively of index given by the topology of the normal bundle in TX over the boundary of Y).

I will explain two ways to ensure smoothness of the moduli space of local associative deformations of Y. The first way is to assume metric conditions on Y and the second is to perturb the G_2 structure in the realm of closed G_2 structures (respectively the boundary X).

Speaker: **Grigorian, Sergey (Stony Brook University)**
Title: **Deformations of G_2-structures with torsion**
Abstract: We consider non-infinitesimal deformations of G_2-structures on 7-dimensional manifolds and derive a closed expression for the torsion of the deformed G_2-structure. We then specialize to the case where the deformation lies in the 7-dimensional representation of G_2 and is hence defined by a vector v. In this case, we explicitly derive the expressions for the different torsion components of the new G_2-structure in terms of the old torsion components and derivatives of v. In particular this gives a set of differential equations for the vector v which have to be satisfied for a transition between G_2-structures with particular torsions. For some specific torsion classes we then explore the solutions of these equations.

Speaker: **Harland, Derek (Loughborough University)**
Title: **Instantons and Killing spinors**
Abstract: I will present some new examples of instantons on manifolds with real Killing spinors and their cones. Examples of manifolds admitting real Killing spinors include nearly Kähler 6-manifolds, nearly parallel G_2-manifolds in dimension 7, Sasaki-Einstein manifolds, and 3-Sasakian manifolds. For each of these classes of manifolds, I will exhibit a connection on the tangent bundle which has reduced holonomy and which solves the appropriate instanton equation. I will also discuss new 1-parameter families of instantons on the cones over real Killing spinor manifolds: these generalise various examples that appeared in the physics literature, and can be lifted to solutions of heterotic supergravity.

Speaker: **Ivey, Thomas (College of Charleston)**
Title: **Austere submanifolds in complex projective space**
Abstract: A submanifold M in Euclidean space $\mathbb{R}^n$ is austere if all odd-degree symmetric polynomials in the eigenvalues of the second fundamental form (in any normal direction) vanish. Harvey and Lawson showed that this condition is necessary and sufficient for the normal bundle of M to be special Lagrangian in $T\mathbb{R}^n \cong \mathbb{C}^n$. A similar result was proved by Karigiannis and Min-Oo for submanifolds in S^n, with TS^n carrying a Calabi-Yau metric due to Stenzel. In this joint work with Marianty Ionel, we determine conditions under which the normal bundle of a *CR*-submanifold in $\mathbb{CP}^n$ is special Lagrangian with respect to the Stenzel metric on $T\mathbb{CP}^n$. We give examples in the case of hypersurfaces in $\mathbb{CP}^2$, and some nonexistence results in the totally real case.

Speaker: **Kovalev, Alexei (University of Cambridge)**
Title: **Asymptotically cylindrical $\mathrm{Spin}(7)$ manifolds**

Abstract: Riemannian manifolds with asymptotically cylindrical ends are essential ingredients in gluing theorems and also have a natural interpretation as having a 'boundary at infinity'. I will report on recent progress in constructing examples of asymptotically cylindrical 8-manifolds with special holonomy $\mathrm{Spin}(7)$. The method uses parts of Joyce's construction of compact $\mathrm{Spin}(7)$ manifolds modified in some important ways and can also be compared at some points with the known constructions of asymptotically cylindrical manifolds with holonomy G_2 and $\mathrm{SU}(n)$.

Speaker: **Lotay, Jason (University College London)**
Title: **Deforming G_2 conifolds**
Abstract: Two natural classes of G_2 manifolds are those which either have non-compact ends asymptotic to cones or have isolated conical singularities. Examples of the former are given by the first complete examples of G_2 manifolds due to Bryant and Salamon, and the latter play an important role in M-Theory. By the fundamental work of Joyce, a compact G_2 manifold M has a smooth moduli space of deformations of dimension $b^3(M)$. I will describe a natural extension of this result to the two aforementioned types of G_2 conifolds. In particular, I will show the stark contrast between the deformation theories in each case and give some applications. This is joint work with S. Karigiannis.

Speaker: **McKay, Benjamin (University College Cork)**
Title: **Some soliton solutions of a flow for G_2-structures**
Abstract: The Laplacian coflow of a G_2-structure is the flow in which the 4-form evolves by its Laplacian. Spiro Karigiannis, Mao-Pei Tsui, and I found a few examples of solitons for this flow.

Speaker: **Mettler, Thomas (Mathematical Sciences Research Institute)**
Title: **Holonomy reduction of 2-Segre structures**
Abstract: The Weyl metrisability problem on a projective surface M corresponds to finding holomorphic curves in a certain quasiholomorphic fibre bundle over M. In this talk I will show that there is a similar correspondence for reducing the holonomy group of a torsion-free 2-Segre structure on an even dimensional manifold.

Speaker: **Pacini, Tommaso (Scuola Normale Superiore)**
Title: **Gluing constructions for special Lagrangian conifolds in $\mathbb{C}^m$**
Abstract: I will present some recent gluing results as in my preprint "Special Lagrangian conifolds, II", available on the arXiv.

Speaker: **Parton, Maurizio (Universita di Chieti-Pescara)**
Title: $\mathrm{Spin}(9)$**, complex structures, and vector fields on spheres**
Abstract: Joint work with Paolo Piccinni and Victor Vuletescu. Although holonomy $\mathrm{Spin}(9)$ appears to be a very restrictive condition, weakened holonomy $\mathrm{Spin}(9)$ conditions have been proposed

and studied in the last years. In this setting, a basic problem is to have a simple algebraic formula for the canonical 8-form Φ whose stabilizer is $\mathrm{Spin}(9)$, as happens for instance in the $\mathrm{Sp(n)}\cdot\mathrm{Sp}(1)$, G_2, and $\mathrm{Spin}(7)$ cases. I will show a nice relation between Φ and a family of almost complex structures J associated to the $\mathrm{Spin}(9)$ structure, leading to an algebraic formula for Φ. I will then show how the existence of more than 7 independent tangent vector fields on spheres is all the fault of $\mathrm{Spin}(9)$, more precisely, all the fault of the J's. Finally, if time permits, the case of metrics which are locally conformal to a parallel $\mathrm{Spin}(9)$ metric will be discussed.

Speaker: **Sá Earp, Henrique (Universidade Estadual de Campinas)**
Title: **Perspectives on G_2-instantons**
Abstract: Solutions to the Hermitian Yang–Mills problem over A. Kovalev's asymptotically cylindrical Calabi-Yau $3-$folds induce instantons over compact $7-$manifolds with holonomy group G_2, obtained by a twisted gluing procedure. Moreover, algebro-geometric monad constructions developed by M. Jardim can be used to generate numerous concrete examples of such G_2-instantons. I will present a survey of that study, punctuated by some open questions ranging from naïve to quite ambitious.

Speaker: **Smith, Aaron (University of Waterloo)**
Title: **A theory of multiholomorphic maps**
Abstract: In recent decades the phenomena associated to pseudoholomorphic curves in Kähler manifolds have led to the discovery of a number of interesting invariants of symplectic manifolds. I will introduce the generalizing framework of multiholomorphic mappings of which the theory of pseudoholomorphic curves forms one of a few families of examples. This is a theory pertaining to mappings (between Riemannian manifolds) which satisfy a particular PDE describing the intertwining of geometric data on domain and target. The higher-dimensional scenario is characterized by a significant amount of rigidity. This will be seen in particular on a family of examples of multiholomorphic maps which involve maps from a 3-manifold into a G_2 manifold. There are close relations to calibrated geometry and mathematical physics.

Speaker: **Wang, Mu-Tao (Columbia University)**
Title: **A separation of variables Ansatz for special Lagrangian submanifolds**
Abstract: I shall discuss new constructions of special Lagrangian submanifolds and self-similar solutions of Lagrangian mean curvature flows based on a separation of variables ansatz. A similar construction for Ricci solitons will also be discussed.

Speaker: **Warren, Micah (Princeton University)**
Title: **Calibrated geometry in the optimal transportation problem**
Abstract: It has been observed that Monge-Ampère equations are related to notions of special Lagrangian submanifolds in manifolds with signature (n, n). We discuss optimal transportation prob-

lems and find that there is a natural metric on the product manifold associated to a given problem. With respect to this metric, the graph of the solution to the optimal transportation problem is a calibrated current. We will say precisely what this means and discuss the many analogies between this setting and the Calabi-Yau setting.

Speaker: **Witt, Frederik (Universität Münster)**
Title: **A variational problem for spinors**
Abstract: We introduce a natural functional on the universal spinor bundle and discuss the Euler-Lagrange equation of the associated variational problem.

Speaker: **Ye, Rugang (University of California at Santa Barbara)**
Title: **The Laplacian flow**
Abstract: The Laplacian flow, introduced by R. Bryant, is a natural evolution equation and serves to deform closed G_2 structures to torsion-free G_2 structures which produce G_2 holonomy. We'll present short-time existence of the Laplacian flow, its stability around torsion-free G_2 structures, long time convergence under a small torsion condition, and the smoothness of the limit map of the Laplacian flow. Additional geometric properties of the Laplacian flow will also be discussed.

4 Informal discussions arising from the meeting

As we hoped, there were a large number of informal discussions, encouraged both by our scheduling large blocks of open time and by the excellent facilities at BIRS. Of course, it is in the nature of informal discussions that we don't have extensive records of them, but what follows is a list of some of the open problems suggested during the informal discussions:

- Regarding the talk of Yunxia Chen, it would be desirable to generalize this work to del Pezzo surfaces. Namely, it would be of interest to construct ADE bundles over a family of del Pezzo surfaces.
- Regarding the talk by Ronan Conlon, it would be very interesting to study the relations of his work to mirror symmetry.
- From the talk by Damien Gayet, could we generalize his work of smoothing the moduli spaces of associative submanifolds in G_2 manifolds to other cases, for instance to complex subvarieties in Kähler manifolds, or to Cayley submanifolds in $\mathrm{Spin}(7)$ manifolds?
- From the talk by Thomas Ivey, it would be interesting to generalize their work to translations of normal bundles.

5 Conclusion

As mentioned above, there were many informal discussions, in particular a great deal of exchanges between experts in different fields. Although it is too early to say exactly what collaborations will arise from this workshop, we are confident that the meeting allowed a significant cross-fertilization between fields, and in particular allowed several of the younger participants to advance their research programme thanks to the advice of more senior scholars.

All of the participants were very enthusiastic about BIRS: the natural setting, the infrastructure and the warmth, hospitality and professionalism of the staff were all very much appreciated.

Selected reading list for special holonomy and calibrations

[1] M. Atiyah and E. Witten, "M-theory dynamics on a manifold of G_2 holonomy", *Adv. Theor. Math. Phys.* **6** (2002), 1–106.

[2] C. Bär, "Real Killing spinors and holonomy", *Comm. Math. Phys.* **154** (1993), 509–521.

[3] A.L. Besse, *Einstein Manifolds*, Springer-Verlag, New York, 1987.

[4] R.L. Bryant, "Some remarks on G_2 structures", *Proceedings of Gökova Geometry-Topology Conference 2005*, 75–109.

[5] R. Harvey and H.B. Lawson, "Calibrated geometries", *Acta Math.* **148** (1982), 47–157.

[6] D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford University Press, 2000.

[7] R. C. McLean, "Deformations of calibrated submanifolds", *Comm. Anal. Geom.* **6** (1998), 705–747.