REPORT ON THE 5-DAY BANFF WORKSHOP 12W502 "STOCHASTIC ANALYSIS & STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS"

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The Banff 5-day workshop 12w5023 was held from Sunday April 1, to Friday April 6, 2012, at the conference site in Banff International Research Station in Alberta. The conference was mainly concerned with the fine analysis of stochastic partial differential equations, hereforth referred to as SPDEs, for the sake of brevity. This topic has experienced tremendous growth, particularly in the past decade for at least two significant reasons: First, a number of central open problems of this area have been solved, and/or are on the verge of being solved; and second, the topic receives continued input from other scientific disciplines, including but not limited to mathematics. Therefore, the organizers felt, and still feel, that the conference was timely.

The main objectives of this meeting were to bring together some of the leading researchers in the analysis of SPDEs, together with select highly-promising young researchers in order to present their recent findings, as well as identify key research problems/areas within the general topic of SPDEs and related fields. The organizers feel strongly that these objectives were met, and that the conference was a success.

The conference was organized roughly as follows: On Monday April 2 through Thursday April 5, the morning talks began at 9:00 a.m. with several half-hour research talks by various leading experts. There was also one 1-hour plenary talk every day in order to introduce to the younger audience aspects of the "big picture" in the modern going-ons of research in SPDEs. These talks were delivered, in chronological order, by Professors Carl Mueller [University of Rochester], Michael Röckner [University of Bielefeld], David Nualart [University of Kansas], and Robert Dalang [Ecole Polytechnique Fédérale de Lausanne]. The evenings of Monday, Tuesday, and Thursday, and all of Friday were dedicated to informal breakout research sessions wherein the audience would spend 5–20 minutes per person describing one, or a series, of open problems and/or directions of modern research interest. These breakout sessions were informal but highly well-attended, and have led to current potential research collaborations among various combinations of the participants. The following is a brief, but still more detailed, synopsis of the lectures at the conference.

Carl Mueller opened the conference with a plenary 1-hour talk on a central open problem for a family of SPDEs that arise in population genetics. The problem is basically the following: Consider the SPDE

$$\frac{\partial}{\partial t}u(t\,,x) = \kappa\frac{\partial^2}{\partial x^2}u(t\,,x) + \rho(u(t\,,x))\,\dot{W}(t\,,x), \tag{1}$$

Date: 4/2/2013.

subject to zero initial data, where $\kappa > 0$ denotes a "viscosity term," $\dot{W}$ denotes space-time white noise, and $\rho(x) = |x|^\alpha$. Clearly, $u(t\,,x) = 0$ is a solution; the question is whether or not $u(t\,,x) = 0$ is the only solution. The particular case that $\alpha = 1/2$ arises most prominently in population genetics, where any solution has the law of the socalled "Brownian density process." Mueller presented his recent non-uniqueness result [joint with L. Mytnik and E. Perkins]: If $\alpha < 3/4$ then the solution is not unique. A major open problem that was introduced is to see whether or not uniqueness holds among all "physical solutions," in this case "all nonnegative solutions." More recently, another participant of this workshop, Mr. Y.-T. Chen, has completed a thesis under the supervision of Professor Edwin Perkins [The University of British Columbia]. Mr. Chen's thesis proves that if we add a nonvoid immigration term to the SPDE (1), then the solution is not unique even among non-negative solutions.

Wenbo Li's lecture gave a bird's-eye view of a number of recent developments in the general theory of "small-value probabilities." A number of novel connections of this topic to SPDEs, branching processes, tauberian theory, Gaussian processes, statistical mechanics [Edwards model] etc. were pointed out. Roughly speaking, the area of small-value probabilities is concerned with the asymptotic behavior of probabilities of the type $\mathrm{P}\{W \le \epsilon\}$ as $\epsilon \downarrow 0$, where W is an interesting non-negative random variable. Among other things, Li introduced his striking work [joint with Q.-M. Shao] on the d-parameter Brownian sheet W: There exist universal constants c_1 and c_2 such that for all $\epsilon > 0$ sufficiently small,

$$\exp\left(-c_1 \left|\log(1/\epsilon)\right|^d\right) \le \mathrm{P}\left\{\sup_{(s,t)\in[0,1]^d} W(s\,,t) < \epsilon\right\} \le \exp\left(-c_2 \left|\log(1/\epsilon)\right|^d\right).$$

A vast array of open problems were presented.

Daniel Conus presented his research on intermittency and chaos for various stochastic systems. In particular, he presented his work [joint with M. Joseph & D. Khoshnevisan] which shows that the solution to the stochastic heat equation (1) with $\alpha = 1$ and initial data [say] $u(0\,,x) \equiv 1$—this is the socalled parabolic Anderson model of mathematical physics—has the following "KPZ scaling property": There exist positive and finite universal constants c_1 and c_2—depending only on the time variable $t > 0$—such that almost surely for all R large,

$$\exp\left(c_1 \frac{(\log R)^{2/3}}{\kappa^{1/3}}\right) \le \sup_{|x|<R} u(t\,,x) \le \exp\left(c_2 \frac{(\log R)^{2/3}}{\kappa^{1/3}}\right).$$

Sandra Cerrai lectured on her work [joint in part with M. Friedlin] on 2-D stochastic Hamiltonian systems of the type

$$\mu \frac{\partial^2}{\partial t^2} q^\mu(t) = b\left(q^\mu(t)\right) + A_0 \frac{\partial}{\partial t} q^\mu(t) + \sigma\left(q^\mu(t)\right) \dot{W}(t),$$

subject to $q^\mu(0) := q \in \mathbf{R}^2$ and $\partial q^\mu(t)/\partial t := p \in \mathbf{R}^2$. Here, $\mu > 0$ is a positive parameter and $\dot{W}$ denotes white noise on $[0\,,\infty)$. When the the real parts of the eigenvalues of A_0 are strictly negative, then the preceding is a generalized 2-D Langevin equation [$\partial q^\mu/\partial t :=$ friction], and Friedlin [2005; also, Chen] have shown that the following Kramers–Smoluchowski approximation is valid for every $T, k > 0$:

$$\lim_{\mu\downarrow 0} \mathrm{E}\left(\sup_{t\le T} |q^\mu(t) - q(t)|^k\right) = 0,$$

where q solves the stochastic differential equation $\mathrm{d}q = (b \circ q)\,\mathrm{d}t + (\sigma \circ q)\,\dot{W}$. Cerrai's talk addressed the remaining case which corresponds to when the Hamiltonian system is describing *charged* particles in a magnetic field. She showed how the Hamiltonian system needs to be regularized, in that case, and presented a homogenization theorem for the regularized equation. As a consequence of this development, Cerrai showed a type of "propagation of chaos" result.

Raluca Balan's lecture revolved around her work on novel linear SPDEs that are driven by fractional noises. An example of her work is the following: Consider the SPDE

$$\frac{\partial}{\partial t} u(t\,,x) = (\mathcal{L}u)(t\,,x) + \dot{W}(t\,,x), \tag{2}$$

with zero initial data, where $\mathcal{L} := -(-\Delta)^{\beta/2}$ denotes the fractional Laplacian of order $\beta/2$, and $\dot{W}$ is a Gaussian noise that is white in space, and whose temporal covariance kernel is of a fractional Brownian motion type,

$$R_H(t\,,s) := H(2H-1)\int_0^t \mathrm{d}u \int_0^s \mathrm{d}v\ |u-v|^{2H-2}. \tag{3}$$

Balan presented a necessary and sufficient condition for the existence of a random field solution to (2) [joint with C. Tudor]. These works were shown to be connected to a new sort of probabilistic potential theory, in particular, to weighted local times functionals

$$L_t(\varphi) := \beta \int_0^t \mathrm{d}r \int_0^t \mathrm{d}s\ |r-s|^{2H-2}\varphi(X_r - Y_s),$$

where X and Y are two independent β-stable Lévy processes. An important ingredient of the existence proof rested on developing a new maximum principle for a corresponding weighted potential kernel. Related results on hyperbolic equations were also presented.

John Walsh's lecture closed the Monday lecture sessions, and contained a new method for the numerical analysis of linear SPDEs of the type,

$$\frac{\partial}{\partial t} u(t\,,x) = \frac{\partial^2}{\partial x^2} u(t\,,x) + \dot{W}(t\,,x), \tag{4}$$

where $\dot{W}$ denotes space-time white noise on $[0\,,1] \times \mathbf{R}$, and the SPDE has a "nice" initial value. It is well known that solution is Hölder continuous of any order $< 1/2$ in the x variable and any order $< 1/4$ in the t variable.

If one discretizes (4)—with respective mesh sizes Δt and Δx for the time and space variables—then the universal error-rates of Davies and Gaines show that the error rate in the resulting numerical scheme is of rough order $\max\{(\Delta t)^{1/4}\,, (\Delta x)^{1/2}\}$. It follows that we need to adopt $\Delta t \approx (\Delta x)^2$ for best results. Walsh's lecture was concerned about the practical problem of making more precise the meaning of "$\approx$" in the preceding discussion. In other words, Walsh proved that there typically is a canonical choice of $c > 0$ such that $\Delta t = c(\Delta x)^2$ is optimal; the value of c depends, among other things, on the particulars of the numerical method that is being used.

For instance, when one applies the Crank–Nicholson numerical scheme for solving (4), then Walsh's striking optimal choice is $\Delta t = (\Delta x)^2/(\pi - 2)$. A series of open problems were also introduced.

Michael Röckner's was the second 1-hour plenary talk of the conference. His lecture was on the regularization of ODEs and PDEs by noise [joint with V. Bogachev, G. Da Prato, N.V. Krylov, E. Priola, and S. Shaposhnikov]. As an example of this general theory, Röckner presented the following ∞-dimensional SDE/SPDE: Let H be a separable Hilbert space, $B : H \to H$ a nice vector field, and $\sigma : H \to L(H)$ measurable. Then, consider the stochastic differential equation,

$$\mathrm{d}X_t^x = B(X_t^x)\,\mathrm{d}t + \sigma(X_t^x)\,\mathrm{d}W_t^x,$$

subject to $W_0^x := x \in H$. Then, it was shown that $p_t f(x) := \mathrm{E}f(X_t^x)$ solves the Fokker–Planck equation, $(\mathrm{d}/\mathrm{d}t)\,p_t f(x) = p_t(Lf)$, and the operator L has a second-order part [because of the noise] which has a regularizing effect on the Fokker–Planck equation. Underlying this theory lies a new method of characteristics for PDEs, which now involves also ideas from the Itô calculus. This method is shown to lead to a uniqueness theorem for the Fokker–Planck equation. Röckner went on to show how to extend this theory in order to establish the existence of pathwise solutions to various infinite-dimensional SDEs that are driven by a "large" white-noise forcing term.

Yaozhong Hu presented a novel Feynman–Kac representation for the parabolic Anderson model (1) [joint with D. Nualart and J. Song], where $\dot{W}$ now denotes a Gaussian noise with covariance form

$$\mathrm{E}\dot{W}(t\,,x)\dot{W}(s\,,y) = R_{H_0}(s\,,t)\cdot\prod_{j=1}^{d} R_{H_i}(x_i\,,y_i),$$

where R_H was defined in (3). Hu's talk established exactly when this SPDE has a [Stratonovich] solution, and that when there is a solution, it has a Feynman–Kac representation. The sufficient condition for the existence and uniqueness of a weak solution was shown to be: $H_0,\ldots,H_d > 1/2$; and $2H_0 + \sum_{i=1}^d H_i > d+1$. In that case, the solution exists, $u(t\,,x)$ has a finite moment generating function near the origin, $(t\,,x) \mapsto u(t\,,x)$ is Hölder continuous a.s., and the law of $u(t\,,x)$ is absolutely continuous with respect to the Lebesgue measure. In the case that $\mathrm{E}\dot{W}(t\,,x)\dot{W}(s\,,y) = R_H(s\,,t)Q(x\,,y)$ for a bounded, jointly Lipschitz continuous Q, and $1/4 < H < 1/2$, Hu showed the existence of a Feynman–Kac formula.

Le Chen's lecture was on stochastic heat equations of the type (1), where ρ is Lipschitz and satisfies the lower cone condition $\inf_x |\rho(x)/x| > 0$. Motivated by statistical mechanics [where $\rho(x) \propto x$ and $u(0\,,x) = \delta_0(x)$], Chen showed his analysis of the preceding SPDE [joint with R. Dalang], wherein they prove that the preceding SPDE has a unique random-field solution provided that $u(0\,,\cdot)$ is a tempered signed measure.

Chen also presented a solution to an open problem of Conus and Khoshnevisan (2012) about the existence of L^2-*intermittency fronts* of the solution. An example of his general theorem is the following: Consider the parabolic Anderson model wherein $\rho(x) = \lambda x$ for some $\lambda > 0$. Then, the lower and upper L^2-intermittency fronts agree and are equal to $\lambda^2/2$; more precisely, for all $\alpha > \lambda^2/2$ and $\beta < \lambda^2/2$,

$$\liminf_{t\to\infty} \frac{1}{t} \sup_{|x|>\alpha t} \log \mathrm{E}\left(|u(t\,,x)|^2\right) < 0, \quad \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|>\beta t} \log \mathrm{E}\left(|u(t\,,x)|^2\right) > 0.$$

This material was borrowed from Mr. Chen's PhD thesis under the supervision of Professor R. Dalang [Ecole Polytechnique Fédérale de Lausanne], and has now been completed.

Leonid Mytnik's lecture was about his work on the multifractal analysis of 1-dimensional super processes with β-stable branching. Let $\{X_t\}_{t\geq 0}$ denote a $(2\,,d\,,\beta)$ super process with β-stable branching, in the sense of Dawson, Perkins, ... Mytnik considers the case that $d = 1$ and $\beta < 1$; this is precisely when $t \mapsto X_t$ is a discontinuous measure-valued process with pure jumps of the form $r\delta_x$, and yet $X_t(\mathrm{d}x) \ll \mathrm{d}x$ a.s. In that case, we can write $X_t(x)$ for $X_t(\mathrm{d}x)/\mathrm{d}x$, and $X_t(x)$ solves the SPDE

$$\frac{\partial}{\partial t} X_t(x) = \frac{\partial^2}{\partial x^2} X_t(x) + [X_{t-}(x)]^{1/(1+\beta)}\, \dot{L}(t\,,x),$$

where $\dot{L}$ denotes a $(1+\beta)$-stable Lévy noise with no negative jumps [Mytnik, 2002]. Perkins and Mytnik (2003) proved that X_t is a.s. continuous for all $t > 0$ fixed, and Fleischmann and Wachtel (2010) established that whenever $0 < \eta < \eta_c := 2/(\beta+1) - 1$, X_t is a.s. Hölder continuous of order η, and that η_c is an optimal choice. Finally, Mytnik introduced two novel results. The first [joint with Fleischmann and Wachtel] shows that if x and $t > 0$ are fixed then the optimal Hölder exponent at x is

$$\bar{\eta}_c := \min\left[\eta_c + \frac{1}{1+\beta}\ ,\ 1\right].$$

For his second main theorem, Mytnik defined the space $C^\eta(x)$ as the space of all functions $f : \mathbf{R} \to \mathbf{R}$ for which we can find a finite constant c and a polynomial P_x of degree $\leq \lfloor \eta \rfloor$ such that $|f(y) - P_x(y)| \leq c|x-y|^\eta$, globally. Define $H(x) := \sup\{\eta : X_t \in C^\eta(x)\}$ as a new measure of optimal Hölder regularity. Then Mytnik showed that if $\eta < 1$ and X_0 is a finite measure, then the measure X_t has the following multifractal behavior: For every open set $U \subset \mathbf{R}$ and $\eta \in [\eta_c\,, \bar{\eta}_c]$,

$$\dim_{\mathrm{H}} \{x \in U :\ H(x) = \eta\} = (\beta+1)(\eta - \eta_c),$$

a.s. on $\{X_t(U) > 0\}$.

Martina Zähle's lecture presented SPDEs driven by gradient noises of the type

$$\partial_t u = -\mathcal{A}^\theta u + F(u) + G(u) \cdot \dot{Z},$$

where $\mathcal{A}$ is the generator of an ultra-contractive semigroup $\{P_t\}_{t\geq 0}$ that has positive and finite spectral dimension, $\theta \leq 1$, and $\dot{G}$ is an arbitrary [possibly random] element of the function space $C^{1-\alpha}([0\,,T]\,, H_2^{\theta\beta}(\mu)^*)$. The main result of this lecture is that the preceding SPDE has a pathwise unique mild solution in the Sobolev space $W^\gamma([0\,,T]\,, H_2^{\theta\delta}(\mu))$ for a suitable choice of $\delta > 0$ [joint with M. Hinz and E. Issoglio].

Franceso Russo discussed his recent work on stochastic and non-stochastic porous media equations [joint with V. Barbu, M. Röckner, N. Belaribi, and F. Cuvelier]. These are models of self-organized criticality that are supposed to describe, for example, the evolution of snow flakes. The general form of the model is

$$\frac{\partial}{\partial t} u(t\,,x) = \frac{1}{2}\Delta(\beta(u)) + u(t\,,x)\dot{W}(t\,,x), \tag{5}$$

where $\dot{W}$ denotes a Gaussian noise that is white in time and possibly nice in x. In the non-random setting, one obtains the PDE $\partial_t u(t\,,x) = \frac{1}{2}\Delta(\beta(u))$, which is supposed to be solved in $L^1(\mathbf{R}^d)$ for example because it is known the the solution, if any, can have discontinuities when the function β is large. An important example is when $\beta(u) = uH(u - e_c)$ for a nice function H and a "critical parameter" e_c. Russo presented a theorem that states that the preceding PDE has a weak solution when β is continuous. Moreover, that solution can be characterized as degenerate [versus non-degenerate] if and only if a certain explicitly-defined function $\Phi = \Phi_\beta$ vanishes at zero. Furthermore, there is a corresponding Fokker–Planck equation. Relations to the stochastic PDE with multiplicative noise were also introduced. Most remarkably, it was pointed out that the stochastic problem hinges on a stochastic Fokker–Planck equation that is of independent interest.

Martina Hoffmanová introduced the main findings of her PhD thesis [under the supervision of Professor A. Debussche] which has quite recently been approved and completed. This talk's main results are on wellposedness problems for kinetic solutions to degenerate parabolic SPDEs such as

$$\mathrm{d}u + \mathrm{div}(B(u))\,\mathrm{d}t = \mathrm{div}(A(x)\nabla u)\,\mathrm{d}t + \Phi(u)\,\mathrm{d}W,$$

where W is a cylindrical Brownian motion in a separable Hilbert space H, $A : \mathbf{T}^n \to \mathbf{R}^{N\times N}$ is a smooth and symmetric positive semidefinite matrix, $B : \mathbf{R} \to \mathbf{R}^N$ is a C^1 flux function of at-most polynomial growth, and $\Phi(z) : H \to L^2(\mathbf{T}^N)$ has linear growth and is coordinatewise Lipschitz for every $z \in L^2(\mathbf{T}^N)$. Using related non-random conservation laws, Hoffmanová introduced a notion of a *kinetic solution*, and went on to prove that if the initial function u_0 is in $L^p(\Omega\,, L^p(\mathbf{T}^N))$, then the degenerate SPDE (5) has a unique kinetic solution that is continuous in its initial data.

Xia Chen studied the stationary parabolic Anderson model,

$$\frac{\partial}{\partial t} u(t\,,x) = \frac{1}{2}\Delta u(t\,,x) + u(t\,,x)V(x),$$

for $x \in \mathbf{R}^d$ and $t \geq 0$, subject to $u(0\,,x) = 1$, where the random potential V has any one of the following four types: (i) V is a stationary Gaussian process with mean zero and a bounded and continuous covariance function; (ii) V is fractional white noise; that is, $V(x) = \partial^d W^H(x)/\partial x_1 \cdots \partial x_d$, for a fractional Brownian motion W^H with Hurst vector $H := (H_1\,, \ldots, H_d)$; (iii) $\{V(x)\}_{x\in\mathbf{Z}^d}$ is a spatial white noise; and (iv) V has the following representation in terms of a white noise W on $\mathbf{R}^d$: $V(x) = \int_{\mathbf{R}^d} \|y - x\|^{-p}\, W(\mathrm{d}y)$.

For case (i) it had been conjectured by Carmona and Molchanov that if the spectral density of V at x behaves as $C/\|x\|^\alpha$ as $\|x\| \to 0$, for some $0 < \alpha < 2 \wedge d$, then $\log u(t\,,0) \sim Ct(\log t)^{(4-\alpha)/(2-\alpha)}$ as $t \to \infty$. Chen shows that this conjecture is incorrect, and can be corrected in the following form:

$$\log u(t\,,0) \sim \kappa(\alpha)[c(d\,,\alpha)]^{2/(4-\alpha)} \cdot t(\log t)^{2/(2-\alpha)} \qquad \text{as } t \to \infty, \tag{6}$$

where $\kappa(\alpha)$ is an explicitly-defined numerical quantity that depends solely on α, and $c(d\,,\alpha)$ is the optimal constant in the following Sobolev-type inequality:

$$\iint\limits_{\mathbf{R}^d\times\mathbf{R}^d} \frac{[f(x)f(y)]^2}{\|x-y\|^\alpha}\,\mathrm{d}x\,\mathrm{d}y \le c(d\,,\alpha)\cdot\|f\|_2^{4-\alpha}\|\nabla f\|_2^\alpha,$$

valid for all $f \in W^{1,2}(\mathbf{R}^d)$. Similar results are shown for cases (ii) and (iv). In both of these cases, (6) holds with the respective choices $\alpha := 2p - 2\sum_{j=1}^d H_j$ [for (ii)] and $\alpha = 2p - d$ [for (iv)]. Finally, Chen showed that in case (iii) [white noise], (6) has to be adjusted as follows:

$$\log u(t\,,0) \sim \frac{1}{2}\left(\frac{3}{2}\right)^{2/3} t(\log t)^{2/3} \qquad \text{as } t \to \infty.$$

In all cases, the proofs involved a delicate large-deviations analysis of the Feynman–Kac formula, $u(t\,,0) = \mathrm{E}(\exp\{\int_0^t V(B_s)\,\mathrm{d}s\}\mid V)$, where B denotes a Brownian motion.

David Nualart delivered the third 1-hour plenary talk of the workshop. Nualart's lecture began with a brief overview of the Malliavin calculus, and in particular, the Nourdin–Viens formula for the density of elements of Malliavin's probabilistic Sobolev space $\mathbf{D}^{1,2}$. Nualart's lecture then proceeded by showing how one can apply these ideas from Malliavin's calculus in concrete problems of SPDEs. As a first example, Nualart presented [a more general form of] the following theorem: Suppose u solves (1) with nice initial data, where $\dot W$ is white in time and spatially homogeneously correlated with a spectral density f that satisfies $\int_{\mathbf{R}^d}(1+z^2)^{-\eta} f(z)\,\mathrm{d}z < \infty$ for some $\eta \in (0\,,3/4)$, then $u(t\,,x)$ has an absolutely-continuous distribution with a density function that satisfies the heat-kernel bounds

$$\frac{c_0}{t^{1-\eta}}\exp\left(-\frac{-(z-m)^2}{c_1t}\right) \le p(z) \le \frac{c_2}{t}\exp\left(-\frac{-(z-m)^2}{c_3t}\right).$$

Next, Nualart discusses degenerate SPDEs of the form

$$\mathcal{H}X_t(x) = -\int_{-\infty}^\infty \frac{\partial}{\partial x}\left(h(y-x)X_t(x)\right)\frac{\partial^2}{\partial t\partial x}W(t\,,y)\,\mathrm{d}y + \sqrt{X_t(x)}\frac{\partial^2}{\partial t\partial x}V(t\,,x),$$

where $\mathcal{H} := \partial_t - \partial^2_{xx}$ denotes the heat operator W and V are independent Brownian sheets, and $h \neq 0$ is a nice function. Such SPDEs arise as continuum limits of interacting particle systems [Dawson, Vaillancourt, Wang, ...]. An open question in this area is to establish the Hölder continuity of the solution $X_t(x)$. Recently (2011) Li, Wang, Xiong, and Zhou have proved that the solution is Hölder continuous for any index $< 1/10$ in t and $< 1/2$ in x. Nualart showed how one can use the Malliavin calculus in a clever way in order to establish the desired Hölder continuity in t of an arbitrary index $< 1/4$ [joint work with Lu and Hu].

Frederi Viens began his lecture with an accessible introduction to the Malliavin calculus and Stein's equation. He then showed how to obtain the following remarkable inequality: If $X \in \mathbf{D}^{1,2}$ is an otherwise arbitrary centered random variable and Z is a standard normal random variable, then

$$\sup_{z\in\mathbf{R}} |\mathrm{P}\{X > z\} - \mathrm{P}\{Z > z\}| \le \mathrm{E}\left(|1 - G_X|\right), \tag{7}$$

where G_X is the random variable that is defined uniquely via the Nourdin–Peccati integration by parts formula $\mathrm{E}(X\varphi(X)) = \mathrm{E}(\varphi'(X)G_X)$, valid for all smooth and bounded functions φ. Viens then showed how one can apply the inequality (7) to the study of a family of continuous 1-dimensional polymer measures. Namely, he considered the parabolic Anderson model

$$\frac{\partial}{\partial t}u(t\,,x) = \frac{\partial^2}{\partial x^2}u(t\,,x) + u(t\,,x)\frac{\partial}{\partial t}W(t\,,x), \tag{8}$$

with $t > 0$ and $x \in \mathbf{R}^d$ and $u(0\,,x) \equiv 1$, where W is a centered Gaussian process with covariance form

$$\mathrm{E}\dot{W}(t\,,x)\dot{W}(s\,,y) = \min(s\,,t)Q(x-y),$$

for a bona fide correlation function Q on $\mathbf{R}^d \times \mathbf{R}^d$ that is bounded [i.e., $Q(0) < \infty$]. The solution to (8) exists and is related to the partition function Z_t^W of a 1-D polymer in random environment W, where $Z_t^W := \mathrm{E}(\exp\{\int_0^t W(\mathrm{d}s\,\mathrm{d}b_s)\}\,|\,W)$, for an independent Brownian motion b. The polymer measure is then $\tilde{P}$, whose Radon–Nikodým derivative is $\mathrm{d}\tilde{P}/\mathrm{dP} = (Z_t^W)^{-1}\exp\{\int_0^t W(\mathrm{d}s\,\mathrm{d}b_s)\}$, where P denotes the Wiener measure. [The "curvilinear stochastic integral" in question is known to exist.] An important question in this area of statistical mechanics is to understand the behavior of the variance of $\log Z_t^W$ as $t \to \infty$.

When $d = 1$ and W is space-time white noise, it is generally believed that $\mathrm{Var}(\log Z_t^W) = t^{(2/3)+o(1)}$ for large t. This conjecture has been verified in recent work by Balazs, Quastel, and Seppäläinen in the case that $u(0\,,x)$ is the exponential of an independent two-sided Brownian motion [this is the invariant measure]. When $Q(0) = 0$—say when $Q(x) = 1 - x^{2\alpha} + o(x^{2\alpha})$ for $x \approx 0$—it is believed that $\mathrm{Var}(\log Z_t^W) = t^{2\chi(\alpha)+o(1)}$ as $t \to \infty$, where $1/3 \le \chi(\alpha) \le 1/2$. Moreoever, it is believed that all values of $\chi(\alpha)$ in this interval are achievable for different models.

Viens proves that if $\inf_{x\in\mathbf{R}^d} Q(x) > 0$, then the conjecture always holds with $\chi(\alpha) = 1/2$. The idea is to write $X := \log Z_t^W$, and estimate the relavant random quantity G_X that arises in (7), for this particular random variable X, using Mehler's formula.

Samy Tindel's lecture concluded the invited research talks of Wednesday, and was concerned with rough SDEs of the type

$$\mathrm{d}Y_t = V_0(Y_t)\,\mathrm{d}t + \sum_{j=1}^d V_j(Y_t)\,\mathrm{d}B_t^j, \tag{9}$$

where the V_i's are bounded and smooth vector fields, and B is a d-dimensional Gaussian process such as fractional Brownian motion [fBm] with Hurst index $1/4 < H < 1/2$. Rough stochastic differential equations of type (9) have been solved recently [Cass, Fritz, Victoir], for example where B is fBm with Hurst index $1/4 < H < 1/2$. However, smoothness has eluded prior attempts. Tindel introduced his recent work [joint with T. Cass, M. Hairer, and C. Litterer] in which they show how the solution to (9) exists and is fairly generally continuous, for instance if B is fBm with $1/4 < H < 1/2$ or if the covariance function R of B satisfies some regularity conditions [enough to ensure strong local non-determinism, for instance]. Some of the key ingredients of the proof were also presented; one particularly noteworthy ingredient was shown to be the following: The Jacobian $\boldsymbol{J}$ that corresponds to (9) itself solves the following rough-path differential equation:

$$\boldsymbol{J}_t = \boldsymbol{I} + \int_0^t DV_0(Y_u)\boldsymbol{J}_u\,\mathrm{d}u + \sum_{j=1}^d \int_0^t DV_j(Y_u)\boldsymbol{J}_u\,\mathrm{d}B_u^j.$$

Robert Dalang began the Thursday lectures with his 1-hour plenary talk on hitting probability estimates for solutions to systems of SPDEs of the form

$$\mathcal{L}u^l = b^l(u) + \sum_{j=1}^d \sigma_{l,j}(u)\dot{W}_j \qquad (1 \le l \le d) \tag{10}$$

where $\mathcal{L}$ acts on the spatial variable $x \in \mathbf{R}^k$ and on the time variable $t \in [0\,,T]$, and $\mathcal{L}$ can denote either the heat operator $\partial_t - \partial^2_{xx}$ or the wave operator $\partial^2_{tt} - \partial^2_{xx}$. The functions b^l and $\sigma_{j,l}$ are Lipschitz continuous, and the initial function u_0 [also u_0' for the wave case] are assumed to be deterministic and given. Finally, the $\dot{W}_j$'s are i.i.d. Gaussian noises; when $k = 1$, they are assumed to be space-time white noise and when $k \ge 2$, they are assumed to be white in time and colored in space according to a Riesz kernel, viz.,

$$\mathrm{E}\dot{W}_i(t\,,x)\dot{W}_j(s\,,y) = \frac{\delta_0(t-s)}{\|x-y\|^\beta} \cdot \delta_{i,j},$$

where $\beta \in (0\,,2 \wedge k)$ in order to ensure the existence and uniqueness of a solution.

The lecture addresses the following question: Given a d-dimensional set A and a hypercube $I \times J \subset (0\,,\infty) \times \mathbf{R}^k$, when is there positive probability that there exists some random $(t\,,x) \in I \times J$ such that $u(t\,,x) \in A$? If such a point exists then we say that A is *nonpolar*; else it is *polar*.

The presented answer depends on whether or not $\mathcal{L}$ is the wave operator or the heat operator. In the case of the wave operator, Dalang showed [joint work with M. Sanz–Solé] in particular that

$$A \text{ is polar if } \dim_{\mathrm{H}}(A) < d - \frac{2+2k}{2-\beta} \text{ and nonpolar if } \dim_{\mathrm{H}}(A) > d - \frac{2+2k}{2-\beta} + \frac{4d^2}{2-\beta}.$$

By contrast, when $\mathcal{L}$ is the heat operator [joint work with D. Khoshnevisan and E. Nualart],

$$A \text{ is polar if } \dim_{\mathrm{H}}(A) < d - \frac{4+2k}{2-\beta} \text{ and nonpolar if } \dim_{\mathrm{H}}(A) > d - \frac{4+2k}{2-\beta}.$$

The proofs hinge on developing detailed "heat-kernel estimates" for random variables of the form $u(t\,,x)$ and $(u(t\,,x)\,,u(s\,,y))$, together with a great deal of Malliavin calculus and probabilistic potential theory, much of which were carefully introduced.

Martin Grothaus presented a lecture on an algebraic SPDE that was derived for a concrete problem in industrial mathematics for textile such as diapers, disposable clothes in hospitals, etc. [in collaboration with the Fraunhofer Institute and others]. Grothaus began his lecture with a detailed description of the underlying problem, and derived a nonlinear SPDE of the form $m\partial^2_{tt}X = \left[\partial_s(\lambda\partial_s X) - b\partial^4_{ssss}X + f(X\,,\partial_t X)\right]\mathrm{d}t + g(X, \partial_t X)\,\mathrm{d}W$, subject to a certain algebraic norm-one condition on $\partial_s X(s\,,t)$. Grothaus showed how one can rewrite a simplified version of the preceding, more succinctly, as an infinite-dimensional SDE of the form

$$\mathrm{d}X(t) = (L(t)X(t) + F(t))\,\mathrm{d}t + G\,\mathrm{d}W(t), \tag{11}$$

subject to $X(t_0) = \xi$, where the operator L leads us to a 2-parameter evolution system $\partial_t U(t\,,\tau)\varphi = L(t)U(t\,,\tau)\varphi$, and the noise has a covariance operator that satisfies $\int_{t_0}^t \mathrm{Tr}\,[U(t\,,r)GQG^*U(t\,,r)^*]\,\mathrm{d}r < \infty$ for all $t \in [t_0\,,T]$. Grothaus then showed [joint work with B. Baur and T. T. Mai] that, under some regularity conditions, (11) has a unique mild solution which is, more significantly, an analytic solution. The preceding does not address the algebraic constraint on the original SPDE. In order to address that matter, Grothaus showed then how J. P. Aubin's work on viability theory can be utilized in the present setting. In order to address that matter, Grothaus showed that a solution to any equation of the form $X'(t) = f(X(t))$ is viable in K—that is, $X(t) \in K$ for all $t_0 \leq t \leq T$—if and only if

$$\liminf_{h\downarrow 0} \frac{\mathrm{dist}(x + hf(x)\,,K)}{h} = 0 \qquad \text{for every } x \in K.$$

Finally, a stochastic version of this result was also briefly mentioned [De Prato and Frankowska]; that result is what is needed in order to build in the algebraic constraints into the original problem.

Leif Döring's lecture revolved around his solution to an old problem in the structure theory of self-similar Markov processes, and its use in the analysis of symbiotic branching processes. Specifically, he presented a complete characterization theorem [joint with M. Barczy] of self-similar Markov processes in terms of a weak solution to a [quite complicated] SDE, thereby also characterizing the solution to the following system that was introduced earlier by Etheridge and Fleischmann:

$$\mathrm{d}u(t\,,k) = \Delta u(t\,,k)\,\mathrm{d}t + \sqrt{\beta u(t\,,k)v(t\,,k)}\,\mathrm{d}B^1_t(k),$$
$$\mathrm{d}v(t\,,k) = \Delta v(t\,,k)\,\mathrm{d}t + \sqrt{\beta u(t\,,k)v(t\,,k)}\,\mathrm{d}B^2_t(k),$$

where B^1 and B^2 denote correlated Brownian motions with $\rho := \mathrm{Corr}(B^1_t\,,B^2_t)$, and the initial states u_0 and v_0 are assumed to be nonnegative.

This model reduces to mutually-catalytic branching process when $\rho = 0$, to the parabolic Anderson model when $\rho = 1$, and to a 2×stepping-stone model when $\rho = -1$. Döring described his work on the behavior of the solution as $\beta \to \infty$: When $\rho = -1$ the solution converges to the voter model; when $\rho = 0$ it converges to a "monster process" [Mytnik and Klenke]; and when $\rho \in (-1\,,1)$, it converges to a "generalized monster process" [joint work with Mytnik]. It was shown how these questions reduce to problems about duality relations. A series of [very hard] open questions were also posed.

Peter Imkeller lectured on his joint work with N. Perkowski, wherein they devise a Fourier-analytic approach of pathwise integration as a possible alternative to other integration theories against rough functions. This idea can be summarized roughly as follows: If $f \in L^2[0\,,1]$, then we can develop f in terms of the Schauder basis $\varphi_n(t) := \int_0^t \chi_n(s)\,\mathrm{d}s$, where the χ_n's are Haar functions. Imkeller showed that whenever f is Hölder continuous of order α, we have a pointwise bound,

$$\left|\sum_{k>K}\sum_{\ell=0}^{2^k-1}\left(\int \chi_{2^k+\ell}\,\mathrm{d}f\right)\varphi_{2^k+\ell}\right| \le C2^{-\alpha K}\|f\|_\alpha,$$

for a universal finite constant C, and all integers $K \ge 1$. Thus, the preceding holds true for all $f \in \mathcal{H} :=$ the closure of C^α with respect to the norm $\|\cdots\|_\alpha$, as well. From this, Imkeller deduced that the map

$$f \mapsto \left(c_n^{-\alpha}\int_0^1 \chi_n\,\mathrm{d}f\right)$$

defines an isomorphism between $\mathcal{H}$ and a sequence space, where the c_n's have a concrete, though somewhat complicated, numerical form. Extensions to other base spaces than C^α were also mentioned, in particular, to Besov spaces $\mathbf{B}^\alpha_{p,q}$.

Finally, an argument was sketched that described how one can plan to construct integrals of the form $\int_0^1 g\,\mathrm{d}f$ for rough functions g and $f \in \mathcal{H}$, using the sequence-space ideas together with methods of rough-path theory.

Jan van Neerven introduced stochastic reaction-diffusion equations of the form

$$\frac{\partial}{\partial t}u(t\,,\xi) = \mathcal{A}u(t\,,\xi) + f(t\,,\xi\,,u(t\,,\xi)) + g(t\,,\xi\,,u(t\,,\xi))\mathcal{R}\dot{W}(t\,,\xi), \tag{12}$$

where the space variable ξ takes values in a bounded open subset O of $\mathbf{R}^d$, $\dot{W}$ dentoes space-time white noise, and $\mathcal{R}$ is a Radonifying separator from $L^2(O)$ to $L^q(O)$ when $d \ge 2$, and $\mathcal{R} :=$ the identity map on $L^2(O)$ when $d = 1$.

Van Neerven addressed the question of global existence of solutions to (12) by rewriting the problem as one about stability of an SDE on a UMD Banach space E:

$$\mathrm{d}X(t) = [\mathcal{A}X(t) + F(t\,,X(t))]\,\mathrm{d}t + G(t\,,W(t))\,\mathrm{d}W(t).$$

Stability theorems were presented that show that if $\mathcal{A}^n \to \mathcal{A}$ in a suitable sense, then the resulting solutions $X^{(n)}$, killed at suitable stopping times, converge to X, killed at a suitable stopping time. And convergence holds in the space $L^0(\Omega\,; B_b([0\,,T]\,;E))$. Moreoever, one can control the behavior of the stopping times well enough to ensure the following result: Under natural regularity assumptions on $\mathcal{A}$, f, if $X_0 \in L^p$ for some p sufficiently large [explicit bounds were shown], then (12) has a global solution that is in $L^p(\Omega\,;C([0\,,T]\times\bar{O}))$ [joint work with M. Kunze].

Łłuís Quer–Sardanyons presented his work [joint with A. Deya and M. Jolis] on the stochastic heat equation. Let W be an $L^2 := L^2[0\,,1]$-valued Brownian motion with [a finite-trace] nuclear covariance Q, and consider the random Stratanovich-type integral equation

$$Y_t = S_t\psi + \int_0^t S_{t-s}\left(f(Y_s)\circ \mathrm{d}W_s\right), \tag{13}$$

where $\{S_t\}_{t\ge0}$ denotes the L^2-semigroup corresponding to $-\Delta$ and $\psi \in L^2$.

Let X denote the mild solution to

$$\mathrm{d}X_t = \Delta X_t\,\mathrm{d}t + V_t^1\,\mathrm{d}t + V_t^2\,\mathrm{d}W_t,$$

where $X_0 = \psi$ and V^i's are continuous-in-L^2 vector fields. Quer–Sardanyons proved that one can always construct the curvilinear stochastic integral $\int_0^t S_{t-s}(f(X_s) \circ \mathrm{d}W_s)$ by first mollifying the white noise and solving the preceding heat equation to obtain X^ϵ—where ϵ is the mollifier parameter–and then letting $\epsilon \downarrow 0$ in order to deduce that $\int_0^t S_{t-s}(f(X_{t-s}) \circ \mathrm{d}W_s) := \lim_{\epsilon\downarrow 0} \int_0^t S_{t-s}(f(X^\epsilon_{t-s}) \circ \mathrm{d}W_s)$ exists in probability. Moreover, that limits was shown to be equal to

$$\int_0^t S_{t-s}\left(f(X_s)\cdot \mathrm{d}W_s\right) + \int_0^t S_{t-s}\left(V_s^2 \cdot f'(X_s)\cdot P\right)\mathrm{d}s,$$

where P is a certain polynomial in the covariance Q, and the first stochastic integral is an Itô integral. Using this stability result, Quer–Sardanyons showed that if $f, f' \in L^\infty$, then (13) has a unique L^2-valued solution. Moreover, Quer–Sardanyons showed that one can use the correlational rough-path analysis of Tindel and Gubinelli (2010) in order to establish that if $f \in C_b^3$ then the solution is Hölder continuous.

Tusheng Zhang's lecture presented a uniqueness theorem for the invariant measure of SPDEs with two reflecting walls [joint work with J. Yang]. Specifically, Zhang considered an SPDE of the following form.

Let h^1 and h^2 denote two reflecting walls on the state space S^1 with the properties that: (H1) $h^1(x) < h^2(x)$ for all $x \in S^1$; and (H2) $\partial^2_{xx} h^j \in L^2(S^1)$ for $j = 1, 2$. The SPDE that was studied seeks to find a random function $u(t\,,x)$ such that $h^1(x) \le u(t\,,x) \le h^2(x)$ for $t > 0$ and $x \in S^1$, and u satisfies

$$\frac{\partial}{\partial t}u(t\,,x) = \frac{\partial^2}{\partial x^2}u(t\,,x) + f(u(t\,,x)) + \sigma(u(t\,,x))\,\dot W(t\,,x) + \eta - \xi, \tag{14}$$

where $f, g : S^1 \to \mathbf{R}$ are Lipschitz continuous, $u(0\,,\cdot) \in C(S^1)$, $\dot W$ denotes space-time white noise, and η and ξ are random measures that are a part of the solution and satisfy $\int_{\mathbf{R}_+\times S^1}(u - h^1)\,\mathrm{d}\eta = \int_{\mathbf{R}_+\times S^1}(h^2 - u)\,\mathrm{d}\xi = 0$. In the one-sided case (say, $h_1 \equiv 0$ and $h_2 \equiv \infty$) the preceding becomes an SPDE with reflection. In that case, when the noise is additive, Nualart and Pardoux have shown that there exists a unique solution; and in the multiplicative case when σ is non-constant Donati Martin and Pardoux established existence, and Xu and Zhang proved uniqueness.

Zhang's lecture presented an argument based on the respective theories of Krylov-Bogolyubov [for existence] and Mueller [for uniqueness] in order to prove that, under the stated assumptions, (14) always has a unique invariant measure.

Annie Millet's lecture on the stochastic Cahn–Hilliard and Allan–Cahn equations concluded the research talks of the conference. As a sampler of the theory presented in this talk, let us consider a nice convex domain $O \subset \mathbf{R}^d$ with piecewise smooth boundary, and denote by ν its outward normal vector. Millet introduced the SPDE

$$\partial_t u = -\rho\Delta\left(\Delta u - f(u)\right) + \left(\Delta u - f(u)\right) + \sigma(u)\dot W, \tag{15}$$

for $(t\,,x) \in [0\,,T]\times O$, subject to

$$\frac{\partial}{\partial \nu}u = \frac{\partial}{\partial \nu}\Delta u = 0 \qquad \text{on } [0\,,T]\times\partial O.$$

The function f is assumed to be a third-degree polynomial with positive leading coefficient; for instance, $f = F'$, where $F(u) = (1-u^2)^2$ denotes the free energy for a double-well potential. Then Millet showed that if $u_0 \in L^q(O)$ for some $q \geq 6$ and σ is Lipschitz continuous with $|\sigma(u)| = O(|u|^\alpha)$ as $|u| \to \infty$ for some $\alpha \in (0\,, 1/9)$, then for all $T > 0$, (15) admits a unique pathwise solution $u \in L^\infty([0\,,T]; L^q(O))$ [joint work with A. Antonopoulos and G. D. Karali]. Similar results were presented for the stochastic Cahn–Hilliard equation.

In the case that $O = (0\,,\pi)^d$ is the open torus, then more information about the solution is available. For example: (i) If u_0 is continuous, then so is u; (ii) If $u_0 \in C^\gamma$ for some $\gamma \in (0\,,1)$, then u is Hölder continuous in its space variable; (iii) If $d = 1, 2, 3$, u_0 is continuous, and $|\sigma(x)| > 0$ for all $x \in O$, then the law of $u(t\,,x)$ is absolutely continuous for all $t > 0$ and $x \in O$.