Lie algebras, torsors and cohomological invariants

Stefan Gille (University of Alberta), Nikita Karpenko (Paris 6),
Arturo Pianzola (University of Alberta), Vera Serganova (University of California, Berkeley),
Kirill Zainoulline (University of Ottawa)

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1 Overview of the Field

The theory of Lie algebras deals with the study and classification of (in-)finite dimensional Lie algebras and has many applications in representation theory, combinatorics and theoretical physics. Many interesting infinite dimensional Lie algebras can be thought as being "finite dimensional" when viewed, not as algebras over the given base field, but rather as algebras over their centroids. From this point of view, the algebras in question look like twisted forms of simpler objects. The quintessential example of this type of behaviour is given by the celebrated affine Kac-Moody Lie algebras which have particular importance in theoretical physics, especially conformal field theory and the theory of exactly solvable models. The connection between the "forms" point of view and Extended Affine Lie Algebras (EALAs for short) – a class of infinite dimensional Lie algebras that as rough approximations can be thought as higher analogues of the affine Kac-Moody Lie algebras – was one of the central themes of the workshop.

The theory of torsors and the associated linear algebraic groups was brought to the forefront of modern algebra by two fundamental discoveries made during the end of the 90's. The first is the proof of Milnor's conjecture by V. Voevodsky [Fields Medal, 2002] which was based on the computation of motivic cohomology of the norm quadric. This inspired an intensive study of quadratic forms, e.g. torsors of an orthogonal group, their motives and cohomological invariants (Karpenko's ICM 2010 lecture). The second discovery is due to Z. Reichstein and deals with the notions of essential and canonical dimensions of linear algebraic groups (Reichstein's ICM 2010 lecture). Roughly speaking, this numerical invariant characterizes the complexity (splitting properties) of a torsor. There are several classical open conjectures in modern algebraic geometry which are closely related to torsors. These formed another central theme of the workshop.

Using cohomological language, a torsor can be identified with a twisted form of an algebraic variety corresponding to a cocycle in the first non-abelian Galois cohomology. This algebraic variety is usually described by some combinatorial data involving root systems and representations, hence, providing a strong connection between the theory of Lie algebras and torsors. The main purpose of the workshop was to exploit and to develop this connection. Namely, using the language of torsors we intended to provide new applications to the theory of Lie algebras; and, vice versa, having in hands Lie algebras to obtain new results concerning various conjectures on torsors and linear algebraic groups.

2 Recent Developments

The last five years can be characterized as a boom of research activity in the workshop areas. To support this observation we should mention the recent results by

(a) Garibald-Merkurjev-Rost-Serre-Totaro-Zainoulline which set up various connections between cohomological invariants and irreducible representations of Lie algebras, e.g. the Dynkin index;

(b) Karpenko-Merkurjev-Reichstein on essential and canonical dimensions of quadratic forms and linear algebraic groups which essentially used the representation theory.

(c) Petrov-Semenov-Zainoulline on motivic decompositions of projective homogeneous varieties, which are based on V. Kac computations of the Chow groups of compact Lie groups.

(d) Gille-Pianzola, where the classification of multiloop Lie algebras has been related to the classification of torsors over Laurent polynomial rings.

(e) Kac-Lau-Pianzola the remarkable fact that the torsor point of view can also be used to study conformal superalgebras, a fact that lead to the concept of differential conformal superalgebras.

Note also that one of the organizers (Karpenko) was invited to give sectional talk at the International Congress of Mathematicians (2010), one of the speakers (Merkurjev) has received the Cole Prize in Algebra (2012) for his achievements in the theory of essential dimension and another speaker (Parimala) was invited to give a plenary talk at the ICM (2010).

3 Open problems and directions

As the subject of the workshop consists of several emerging areas of modern algebra/algebraic geometry it contains many open questions and problems. These can be described as follows

I. Torsors and cohomological invariants

Cohomological invariants existed long before this terminology was introduced by Jean-Pierre Serre in the mid of 90's. For instance the (signed) determinant and the Clifford algebra of a quadratic form can be considered as a cohomological invariant with values in the Galois cohomology. To bring some order in the various existing invariants Serre developed a theory of cohomological invariants in the following abstract setting:

By a cohomological invariant one means a natural transformation from the first Galois cohomology with coefficients in an algebraic group G (the pointed set which describes all G-torsors) to a cohomology functor $h(-)$, where h is a Galois cohomology with torsion coefficients, a Witt group, a Chow group with coefficients in a Rost cyclic module M, etc. The ideal result here would be to construct enough invariants to classify all G-torsors.

This concept was developed further by Garibaldi-Merkurjev-Rost-Totaro, leading to the complete understanding of invariants in lower degrees. For instance, in degree 2 the group of invariants is generated by the classes of Tits algebras in the Brauer group and in degree 3 it is generated by the Rost invariant.

I.a) Applications to extended affine Lie algebras

The connection between the "forms" point of view, which is related to the theory of Reductive Group Schemes developed by Demazure and Grothendieck, and Extended Affine Lie Algebras (EALAs) is one of the central themes of the proposal. P. Gille and A. Pianzola have pioneered this approach. The language and theory of G-torsors, where G is a reductive group scheme over a Laurent polynomial ring, appears then quite naturally. This point of view brings extremely powerful tools to the study of infinite dimensional Lie theory.

On the other side the study of EALAs has been greatly simplified by work of E. Neher, which reduces the classification problem to the so called cores of the EALA, which are are in fact multiloop algebras (except for a family of fully understood cases). It is not true, however, that every multiloop algebra is the centreless core of an EALA. Finding cohomological invariants that characterize the isomorphism classes of torsors corresponding to the multiloop algebras attached to EALAs is part of the proposal. This question is connected with some deep work in progress by Chernousov, Gille and Pianzola dealing with the problem of conjugacy of Cartan subalgebras of multiloop algebras.

I.b) Applications to representations and cohomology of Lie algebras and homogeneous spaces

The cohomology theory of Lie algebras on one hand is used to compute central and abelian extensions of Lie algebras. Central extensions of Lie algebras often have richer representation theories than their centerless quotients. On the other hand, cohomology of manifolds connects representation theory with geometry and physics. An example of such an interplay is a recently established cohomological interpretation of the elliptic genus, an object originally introduced in string theory.

Another interesting direction is to study maps between cohomologies of vector bundles on homogeneous varieties arising from homomorphisms of the underlying homogeneous varieties. So far the best understood case is the case of varieties of Borel subgroups. In particular, Dimitrov-Roth have showed that the diagonal embedding leads to a natural geometric construction of extreme components of the tensor product of irreducible representations.

II. Torsors and Motives

Following the general philosophy of Grothendieck one can introduce a universal cohomological invariant which takes values in the category of motives. The link between the world of motives and torsors is provided by the celebrated Rost Nilpotence Theorem (RNT) which can be viewed as a generalized Galois descent property. In [Duke 2003] Chernousov-Gille-Merkurjev proved the RNT for arbitrary projective homogeneous varieties over semisimple algebraic groups, hence, opening the door to the study of motives of projective homogeneous varieties.

II.a) Applications to linear algebraic groups

Based on this result and computations of Victor Kac of the Chow ring of G Petrov-Semenov-Zaynullin [Ann. Sci. ENS, 2008] computed the motive of generically split projective homogeneous varieties in terms of generalized Rost motives introduced by Voevodsky. They also showed that the motivic behavior of such varieties can be described by a certain discrete numerical invariant, the J-invariant. As an application of the motivic J-invariant Petrov-Semenov [Duke, 2010] classified all generically split homogeneous varieties of linear algebraic groups. As another application, Semenov (2010) has given a construction of a cohomological invariant of degree 5 of an exceptional group of type E_8, hence, proving a conjecture by J.-P. Serre. This invariant has tremendous applications to the study of subgroups of compact Lie groups of type E_8.

II.b) Applications to quadratic forms and algebras with involutions

Quadratic forms and central simple algebras with involutions provide classical examples of torsors for a (projective) orthogonal group. Its cohomological invariants and motives have been extensively studied during the last decade. We should mention here the works of Karpenko and Vishik who use motives to investigate the splitting behavior of quadratic forms and algebras with involutions, e.g. the Vishik's construction of fields with u-invariant $2^r + 1$, $r > 3$, his motivic decomposition type theory; Karpenko's result on the first Witt indices of quadratic forms and the proof of the Hoffmann's conjecture, his recent proofs of hyperbolicity and isotropy conjectures for algebras with involutions. In the proof of the hyperbolicity conjecture Karpenko (2010) uses the theory of upper motives. Observe that for a projective homogeneous variety the dimension of its upper motive measures its canonical p-dimension, hence, providing a new approach to study the canonical and essential dimensions of algebraic groups.

II.c) Applications to Del Pezzo surfaces and representations of exceptional algebraic groups.

In 1990 Batyrev conjectured that universal torsors over Del Pezzo surfaces can be embedded into homogeneous spaces of exceptional algebraic groups. The particular cases of this conjecture were proven by Popov and Derenthal. Serganova and Skorobogatov recently suggested a universal proof of the Batyrev conjecture which covers a new case of E_8. Very recently Gille [Invent.Math. 2010] has proven the RNT for del Pezzo surfaces, hence, providing a new approach to study their geometry, motives and Chow groups.

4 Presentation Highlights and Scientific Progress Made

I. The first day of the workshop was devoted to general lectures on the topics related to linear algebraic groups and cohomology theories. There were two morning talks by senior researchers E. Bayer-Fluckiger (EPFL, Switzerland) and A. Vishik (Nottingham, UK). The afternoon session was started with the talk by A. Merkurjev (UCLA, USA) - recipient of the Cole prize in Algebra (2012). The last talk of the afternoon was given by the young researcher S. Baek (KAIST, South Korea).

Eva Bayer-Fluckiger *Embeddings of maximal tori of type CM in orthogonal groups*

Let k be an algebraic number fields. Embeddings of maximal tori in orthogonal groups have been studied in several papers, and occur in various arithmetic questions. The case of tori of type CM (that is, tori associated to CM etale algebras) is of special interest of some of the applications. The aim of this talk was to give necessary and sufficient criteria for such an embedding to exist under some conditions, which are fulfilled in the CM case.

Alexander Vishik *Stable and Unstable operations in Algebraic Cobordism*

In the talk the speaker described and effectively constructed (unstable) additive operations $A \to B$, where A is a theory obtained from Algebraic Cobordism of M. Levine - F. Morel by change of coefficients, and B is any Generalized Oriented Cohomology Theory. Among the applications of this technique are the following major results:

1. Description of unstable operations in Algebraic Cobordism theory. The description of stable ones comes as well
2. The Theorem claiming that multiplicative operations $A \to B$ (where A, B are as above) are in 1-to-1 correspondence with the homomorphisms of the respective formal group laws.
3. The construction of Integral (!) Adams operations in Algebraic Cobordism and all the theories obtained from it by change of coefficients (giving classical Adams operations in case of K_0).
4. The construction of Symmetric Operations for all primes p (previously known only for $p = 2$), and the construction of Tom Dieck - style Steenrod operations in Algebraic Cobordism.

Alexander Merkurjev *Generic values of quadratic forms and essential dimension*

Let $f : X \longrightarrow Y$ be dominant map between varieties over the field F. The functor $\mathcal{A}_f$ assigns to a field extension $K \supseteq F$ the image of the induced map $X(K) \to Y(K)$. In this talk the speaker introduced and discussed the essential dimension of the functor $\mathcal{A}_f$ for various f.

In particular he discussed in detail the following example. Let (V, q) be a quadratic space over the field F and V_0 the open subset of all $v \in V$, such that $q(v) \neq 0$. Then by restriction q induces a morphism $V_0 \longrightarrow \mathbb{G}_m$ (also denoted by q).

He showed then using the "general" theory for the functor $\mathcal{A}_q$ the following theorem:

- Let $q = \sum_{i=1}^{n} a_i x_i^2$ be an anisotropic quadratic form over the field F, and $F \subseteq L \subseteq F(x_1, \ldots, x_n)$ be a field which contains the generic value $q = q(x_1, \ldots, x_n)$. If there are $b_1, \ldots, b_n \in L^n$, such that $q(x_1, \ldots, x_n) = q(b_1, \ldots, b_n)$ then the degree $[F(x_1, \ldots x_n) : L]$ is finite and odd.

Sanghoon Baek *On the torsion of Chow groups of Severi-Brauer varieties*

Let p be a prime and A a central simple algebra of p-power degree over a field. We denote by $SB(A)$ the corresponding Severi-Brauer variety. Consider the Grothendieck ring $K(SB(A))$ and its gamma filtration $\Gamma^d K(SB(A))$ for $d \geq 0$. By a theorem of Quillen, the gamma filtration on $K(SB(A))$ is determined by the indices of (tensor) powers of A. Based on this observation, Karpenko introduced the sequence of the exponents of distinct indices of powers of A, which is called the reduced sequence of A. Moreover, Karpenko showed that the torsion part of the 2nd quotient $\Gamma^2 K(SB(A))/\Gamma^3 K(SB(A))$ of the gamma filtration is determined by a certain index of the reduced sequence. Note that the 0th and the 1st quotients are torsion-free. Now we consider the torsion part of Chow group $CH^d(SB(A))$ of cycles modulo the rational equivalence relation. For $d = 0, 1$, they are all torsion free. However, for $d = 2$ there is torsion and it is shown that the torsion part is annihilated by the order of torsion subgroup of $\Gamma^2 K(SB(A))/\Gamma^3 K(SB(A))$. In the talk the speaker provided upper bounds for the annihilators of the torsion subgroups of Chow groups of the Severi-Brauer varieties for a large class of central simple algebras (see [1]).

II. The second day of the workshop was devoted to the interactions between the classical theory of (infinite-dimensional) Lie algebras on one hand side and torsors, linear algebraic groups on the other side. The morning session was started by the talk of V. Popov devoted to the celebrated Gelfand-Kirillov's conjecture. Then E. Neher presented recent developments in the theory of derivations of Lie algebras.

The afternoon session consisted of talks by a graduate student Z. Chang and young researchers N. Lemire and I. Dimitrov.

Vladimir Popov *Rational functions on semisimple Lie algebras and the Gelfand-Kirillov Conjecture*

The talk was aimed at describing the recent solution of the rationality problem for fields of rational functions on semisimple Lie algebras and the intimately related construction of counterexamples to the Gelfand–Kirillov conjecture on the fields of fractions of universal enveloping algebras of simple Lie algebras. This solution exploits a notion generalizing that of the usual torsor.

More precisely, a field extension E/F is called pure (or purely transcendental or rational) if E is generated over F by a finite collection of algebraically independent elements. A field extension E/F is called stably pure (or stably rational) if E is contained in a field L which is pure over both F and E. Finally, we shall say that E/F is unirational if E is contained in a field L which is pure over F.

Let k be a field of characteristic 0. Let G be a connected reductive algebraic group over k. Let V be a finite dimensional k-vector space and let $G \to GL(V)$ be an algebraic group embedding over k. Let $k(V)$ denote the field of k-rational functions on V and $k(V)^G$ the subfield of G-invariants in $k(V)$. It is natural to ask whether $k(V)/k(V)^G$ is pure (or stably pure).

This question may be viewed as a birational counterpart of the classical problem of freeness of the module of (regular) covariants, i.e., the $k[V]^G$-module $k[V]$ (Here $k[V]$ is the algebra of k-regular functions on V and $k[V]^G$ is the subalgebra of its G-invariant elements.) The question of rationality of $k(V)$ over $k(V)^G$ also comes up in connection with counterexamples to the Gelfand-Kirillov conjecture.

Recall that a connected reductive group G is called split if there exists a Borel subgroup B of G defined over k and a maximal torus in B is split. If G is split and the G-action on V is generically free, i.e., the G-stabilizers of the points of a dense open set of V are trivial, then the following conditions are equivalent:

(i) the extension $k(V)/k(V)^G$ is pure;

(ii) the extension $k(V)/k(V)^G$ is unirational;

(iii) the group G is a special group.

Over an algebraically closed field, special groups were defined by Serre and later classified by himself and Grothendieck.

The purity problem for $k(V)/k(V)^G$ is thus primarily of interest in the case where the G-action on V is faithful but not generically free. For k algebraically closed, such actions have been extensively studied and even classified, under the assumption that either the group G or the G-module V is simple.

Let g be the Lie algebra of G. The homomorphism $Int\colon G \to Aut(G)$ sending $g \in G$ to the map $Int(g)\colon G \to G$, $x \mapsto gxg^1$, determines the conjugation action of G on itself, $G \times G \to G$, sending (g, x) to $Int(g)(x)$. The differential of $Int(g)$ at the identity is the linear map $Ad(g)\colon g \to g$. This defines an action of G on g, called the adjoint action. As usual, we will denote the fields of k-rational functions on G, respectively, g, by $k(G)$, respectively, $k(g)$, and the fields of invariant k-rational functions for the conjugation action, respectively, the adjoint action, by $k(G)^G$, respectively, $k(g)^G$.

The purpose of the talk was to address the following purity questions (see [2]):

- Is the field extension $k(g)/k(g)^G$ pure? stably pure?
- Is the field extension $k(G)/k(G)^G$ pure? stably pure?

Erhard Neher *Derivations of algebras obtained by étale descent*

Let g be a simple finite-dimensional Lie algebra over the complex numbers. The celebrated affine Kac-Moody Lie algebras are of the form $E = L \oplus kc \oplus kd$, where L is a (twisted) loop algebra of the form $L(g, \pi)$ for some diagram automorphism π of g. The element c is central and d is a degree derivation for a natural grading of L. It is thus natural to study the derivations of loop and, more generally, multiloop algebras.

The speaker described derivations of Lie algebras obtained by étale descent and discussed various applications to multiloop algebras and extended affine Lie algebras. The talk was based on joint work with Arturo Pianzola [3].

Zhihua Chang *Twisted Loop Algebras Based on Conformal Superalgebras*

Superconformal algebras are infinite dimensional Lie superalgebras of interest in theoretical physics. They are closely related to twisted loop algebras based on conformal superalgebras. To classify twisted loop algebras based on a given conformal superalgebra, differential conformal superalgebras were introduced by V. Kac, M. Lau, and A. Pianzola in 2009. In this talk, the speaker gave first a brief introduction to the general theory of twisted forms of differential conformal superalgebras. After that he discussed the classification of twisted loop algebras based on the $N = 1, 2, 3$, small $N = 4$ and large $N = 4$ conformal superalgebras.

Ivan Dimitrov *Constructing subrepresentations via the Borel-Weil-Bott theorem*

An embedding $G \subset G'$ of reductive algebraic groups gives rise to an embedding $G/B \subset G'/B'$ of the corresponding homogeneous varieties. For any line bundle L' on G'/B' one has the natural map of cohomologies $\pi : H^q(G'/B'; L) \to H^q(G/B, L)$, where L is the restriction of L' to G/B. The Borel-Weil-Bott theorem implies that the dual map π, when nonzero, is a Gmodule homomorphism $\pi : V \to V'$, where V and V' are irreducible modules respectively over G and G'. Varying L' (and respectively q) so that $H^q(G'/B'; L) = (V')^*$ we obtain a purely geometric construction of certain irreducible Gsubmodules of V' which we call cohomological components. In this talk the speaker discussed several types of embeddings $G \subset G'$ and the corresponding cohomological components their properties, relationship to other interesting problems as well as necessary and sufficient conditions for non vanishing of π. In the case when G is embedded diagonally into $G' = G \times G$, the cohomological components lie on faces of the Littlewood-Richardson cone of codimension equal to the rank of G. With an appropriate choice of an embedding of G/B, one can also obtain generators of the algebra of invariant polynomials on the Lie algebra of G as cohomological components. The talk was based on the joint work with Mike Roth as well as results of Valdemar [4].

Nicole Lemire *Stably Cayley Groups over Fields of Characteristic 0.*

Let k be a field of characteristic 0 and G be a connected linear algebraic k-group. We say that a birational isomorphism $\phi\colon G \to Lie(G)$ is a Cayley map if it is equivariant with respect to the conjugation action of G on itself and the adjoint action of G on its Lie algebra $Lie(G)$, respectively. A Cayley map can be thought of as a (partial) algebraic analogue of the exponential map. A prototypical example is the classical Cayley map for the special orthogonal group SO_n defined by A. Cayley in 1846. We say that G is a Cayley group if it admits a Cayley map. We say that G is stably Cayley if $G \times_k \mathbb{G}_m^r$ is Cayley for some $r \geq 0$, where $\mathbb{G}_m$ denotes the multiplicative group. In the case where k is algebraically closed, Cayley and stably Cayley groups were studied by the speaker, Popov and Reichstein.

In this talk the speaker focused on the case where k is an arbitrary field of characteristic 0. The following toy example illustrates how much more intricate the notions of Cayley and stably Cayley group become in this situation.

Let T be a k-torus of dimension d. By definition, T is Cayley (respectively, stably Cayley) over k if and only if T is k-rational (respectively, stably k-rational). If k is algebraically closed, then $T = \mathbb{G}_m^d$, hence T is always rational, and thus always Cayley.

Observe that there is a well-known criterion for stable rationality of T in terms of its character lattice $X(T)$: T is stably rational if and only if the character lattice $X(T)$ is quasi-permutation. Note that the term character lattice here is the lattice of characters of T with the natural action of the absolute Galois group $Gal(k)$. It has been conjectured that every stably rational torus is rational. To the best of our knowledge, this conjecture is still open, and there is no simple lattice-theoretic criterion for the rationality of T.

The talk was based on the recent joint work with Blunk, Borovoi, Kunyavskii and Reichstein (see [5]).

III. The third day of the workshop was devoted to recent trends in the theory of the u-invariant of quadratic forms – an important invariant of torsors for orthogonal groups. This consisted of talks by R. Parimala and D. Saltman.

Raman Parimala *Bounding symbol lengths in Galois cohomology*

Bounding symbol lengths in Galois cohomology has had important implications to bounding the u-invariant of fields. This approach leads to finiteness of the u-invariant of function fields in one variable over a totally imaginary number field, provided a conjecture of Colliot-Thélène on the Brauer-Manin obstruction and the existence of zero cycles of degree one on smooth projective varieties over number fields holds.

David Saltman *Finite u Invariant and Bounds on Cohomology Symbol Lengths*

At a AIM workshop in January 2011, Parimala asked whether in a field with finite u invariant there was a bound on the "symbol length" of any element of μ_2 cohomology in any degree. The speaker answered this question in the affirmative for fields of characteristic 0, and at the same time obtained bounds on the Galois groups that realize all the properties of these cohomology elements and showed that his results extend to finite field extensions.

IV. The morning session of the fourth day of the workshop was devoted to the weak commensurability problem that gives connection between algebraic groups and differential geometry. There were two talks on this subject by S. Garibaldi and V. Chernousov. The afternoon session was devoted to various topics – from the theory of central simple algebras to motives and Schubert calculus in algebraic cobordism. All talks where given by young researchers (M. Florence, O. Haution and V. Kiritchenko).

Skip Garibaldi *Algebraic groups and weak commensurability*

The notion of weak commensurability introduced by Gopal Prasad and Andrei Rapinchuk gives deep connections between algebraic groups and differential geometry, as well as new tools for studying algebraic groups over number fields. The speaker discussed recent applications to the questions:

- If two simple linear algebraic groups over a number field have the same isogeny classes of maximal tori, must the groups be isogenous?
- If two locally symmetric spaces M_1 and M_2 are weakly commensurable – i.e., if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$ where L denotes the set of lengths of closed geodesics—must M_1 and M_2 have a common finite-sheeted cover?

Although there are well-known examples where the answer to each of these questions is "no", the answer is nonetheless frequently "yes".

Vladimir Chernousov *On the genus of a division algebra*

Let K be a field, $Br(K)$ be its Brauer group, and for any integer $n > 1$ let $nBr(K)$ be the subgroup of $Br(K)$ annihilated by n. For a finite-dimensional central simple algebra A over K, we let $[A]$ denote the corresponding class in $Br(K)$, and we then define the genus $gen(D)$ of a central division K-algebra D of degree n to be the set of classes $[D]Br(K)$ where D is a central division K-algebra having the same maximal subfields as D (in more precise terms, this means that D has the same degree n, and a field extension P/K of degree n admits a K-embedding $P \to D$ if and only if it admits a K-embedding $P \to D'$. The speaker addressed the following two questions:

- When does $gen(D)$ consist of a single class ?
- When is $gen(D)$ finite ?

The key result discussed at the talk can be formulated as follows:

Let K be a field of characteristic different from 2. (1) If K satisfies the following property:

(*) if D and D are central division K-algebras of exponent 2 having the same maximal subfields then $D = D$ (in other words, for any D of exponent 2, $|gen(D) \cap 2Br(K)| = 1$),

then the field of rational functions $K(x)$ also satisfies (*). (2) If $|gen(D)| = 1$ for any central division K-algebra D of exponent 2, then the same is true for any central division $K(x)$-algebra of exponent 2.

The talk was based on the joint work with A. Rapinchuk and I. Rapinchuk [6].

Mathieu Florence *Central simple algebras of index p^n in characteristic p*

Let k be a field of characteristic $p > 0$, and let A/k be a central simple algebra of index $d = p^n$ and exponent p^e. Using a result of Hochschild, of which we provide a new proof, we show that A is Brauer equivalent to the tensor product of at most $d - 1$ cyclic algebras of degree p^e. This improves drastically the previously known upper bounds, mainly due to Teichmller, Mammone and Merkurjev.

Olivier Haution *Invariants of upper motives*

The canonical dimension of a smooth complete algebraic variety measures to which extent it can be rationally compressed. In order to compute it, one often rather study the p-local version of this notion, called canonical p-dimension (p is a prime number). In this paper, we consider the relation of p-equivalence between complete varieties, constructed so that p-equivalent varieties have the same canonical p-dimension. In particular, two complete varieties X and Y are p-equivalent, for any p, as soon as there are rational maps $X \to Y$ and $Y \to X$. In order to obtain restrictions on the possible values of the canonical p-dimension of a variety, one is naturally led to study invariants of p-equivalence. In the talk the speaker introduced a systematic way to produce such invariants (and in particular, birational invariants), starting from a homology theory. He provides examples related to K-theory and cycle modules. Then he describes the relation between two such invariants of a complete variety X: its index n_X, and the integer d_X defined as the g.c.d. of the Euler characteristics of the coherent sheaves of $\mathcal{O}_X$-modules. The latter invariant contains both arithmetic and geometric information; this can be used to give bounds on the possible values of the index n_X (an arithmetic invariant) in terms of the geometry of X. For instance, a smooth, complete, geometrically rational (or merely geometrically rationally connected, when k has characteristic zero) variety of dimension $< p1$ always has a closed point of degree prime to p. The talk is based on the paper [7].

Valentina Kiritchenko *Schubert calculus for equivariant algebraic cobordism*

Let k be a field of characteristic zero, and G a connected reductive group split over k. Recall that a smooth spherical variety is a smooth k-scheme X with an action of G and a dense orbit of a Borel subgroup of G. Well-known examples of spherical varieties include flag varieties, toric varieties and wonderful compactifications of symmetric spaces. In her talk, the speaker discussed the equivariant cobordism rings of the following two classes of spherical varieties: the flag varieties and the wonderful symmetric varieties of minimal rank (the latter include wonderful compactifications of semisimple groups of adjoint type). The equivariant cohomology and the equivariant Chow groups of these two classes of spherical varieties have been extensively studied before. Based on the theory of algebraic cobordism by Levine and Morel, and the construction of equivariant Chow groups by Totaro and Edidin-Graham, the equivariant cobordism was initially introduced by D. Desphande for smooth varieties. It was subsequently developed into a complete theory of equivariant oriented Borel-Moore homology for all k-schemes by A. Krishna. Similarly to equivariant cohomology, equivariant cobordism is a powerful tool for computing ordinary cobordism of the varieties with a group action. The techniques of equivariant cobordism have been recently exploited to give explicit descriptions of the ordinary cobordism rings of smooth toric varieties, and that of the flag bundles over smooth schemes. The talk was based on the joint work with Amalendu Krishna [8].

V. The last day of the workshop started with talks on essential dimension by Shane Cernele (graduate student) and Roland Lotscher (young researcher) and was finished by the talk of Philippe Gille devoted to the topological properties of torsors.

Shane Cernele *Essential dimension and error-correcting codes*

Let p be a prime, $r \geq 3$, and $n_i = p^{a_i}$ for positive integers $a_1, \ldots, a_r$. Define $G = GL_{n_1} \times \ldots \times GL_{n_r}$ and let μ be a central subgroup of G, over a field of characteristic zero. The Galois cohomology set $H^1(K, G/\mu)$ classifies r-tuple of central simple algebras satisfying linear equations in the Brauer group $Br(K)$. The key object of the talk is the essential dimension and essential p-dimension of G/μ. To any central subgroup μ, one associates a finite module C called the code associated to μ, and define a weight function w from C to the positive integers. In the talk the speaker showed that the essential dimension of G/μ depends only on C. Using general cohomological methods as well as results of Karpenko and Merkurjev, and Popov, he gave lower and upper bounds on the essential dimension of G/μ in terms of w and C. For some subgroups μ he found matching lower and upper bounds.

Roland Lötscher *Essential p-dimension of algebraic groups, whose connected component is a torus.*
Let p be a prime integer and k a base field of characteristic not p. In this talk the speaker was studying the essential dimension of linear algebraic k-groups G whose connected component G_0 is an algebraic torus. This is a natural class of groups; for example, normalizers of maximal tori in reductive linear algebraic groups are of this form.

To state the main result, recall that a linear representation $\rho\colon G \to GL(V)$ is called generically free if there exists a G-invariant dense open subset UV such that the scheme-theoretic stabilizer of every point of U is trivial. A generically free representation is clearly faithful but the converse does not always hold; see below. We will say that ρ is p-generically free (respectively, p-faithful) if $ker\rho$ is finite of order prime to p, and ρ descends to a generically free (respectively, faithful) representation of $G/ker\rho$.

Then the main result says the following

Let G be an extension of a (possibly twisted) finite p-group F by an algebraic torus T defined over a p-special field k of characteristic not p. Then $\min\dim(\rho) - \dim G \leq ed(G;p) \leq \min\dim(\mu) - \dim G$, where the minima are taken over all p-faithful linear representations ρ of G and p-generically free representations μ of G, respectively.

The talk was based on the joint work with Mark MacDonald, Aurel Meyer and Zinovy Reichstein [9].

Philippe Gille *Topological properties of torsors and homogeneous spaces over valued fields*
This was a report on work in progress with Laurent Moret-Bailly [10]. Let K be the fraction field of a henselian valuation ring R of positive characteristic p. Let Y be a K-variety, H an algebraic group over K, and $f : X \to Y$ an H-torsor over Y. The speaker considered the induced map $X(K) \to Y(K)$, which is continuous for the topologies deduced from the valuation. If Z denotes the image of this map, he investigated the following questions:

(a) Is Z locally closed (resp. closed) in $Y(K)$?

(b) Is the continuous bijection $X(K)/H(K) \to Z$ a homeomorphism ?

5 Outcome of the Meeting

The workshop attracted 42 leading experts and young researchers from Belgium, Canada, France, South Korea, Germany, Russia, Switzerland, USA. There were 19 speakers in total: 10 talks were given by senior speakers, 7 talks by young researches and postdocs and 2 talks by PhD students.

The morning lectures given by senior speakers provided an excellent overview on the current stage of research in the theory of Lie algebras, torsors and cohomological invariants. There were several new results announced, e.g. Merkurjev (on essential dimension), Saltman (on the u-invariant), Vishik (on the cohomological operations). The afternoon sessions provided a unique opportunity for young speakers to present their achievements. Numerous discussions between the participants after the talks have already lead to several joint projects, e.g. Neher-Pianzola, Calmès-Zainoulline.

The organizers consider the workshop to be a great success. The quantity and quality of the students, young researchers and the speakers was exceptional. The enthusiasm of the participants was evidenced by the frequent occurrence of a long line of participants waiting to ask questions to the speakers after each lecture. The organizers feel that the material these participants learned during their time in BIRS will prove to be very valuable in their research and will undoubtedly have a positive impact on the research activity in the area.

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