Sarason Conjecture and the Composition of Paraproducts

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1 Overview of the Research in Teams Project

The project that we worked on during the Research in Teams at Banff was a dyadic version of the Sarason Conjecture. This discrete problem is already very challenging and captures much of the difficulty associated with the conjecture. In particular, we were concerned with dyadic Haar paraproducts, and obtaining necessary and sufficient conditions for the boundedness of the composition of two such paraproducts. The conditions characterizing the boundedness of the composition will be much more general than the boundedness of each individual paraproduct.

Let $\mathcal{D}$ be the dyadic intervals and let $\{h_I \mid I \in \mathcal{D}\}$ be the $L^2(\mathbb{R})$ normalized Haar basis. Set $h_I^0 := h_I$, with the superscript 0 for the mean zero, and set $h_I^1 := |h_I|$, with the superscript 1 for the non-zero mean. Given a function b, and choices $\epsilon, \delta \in \{0,1\}$ define the paraproduct

$$f \in L^2(\mathbb{R}) \mapsto \mathrm{P}_b^{\epsilon,\delta} f(x) := \sum_{I\in\mathcal{D}} \frac{\langle b, h_I\rangle_{L^2}}{\sqrt{|I|}} \left\langle h_I^\delta, f\right\rangle_{L^2} h_I^\epsilon(x).$$

These discrete paraproduct operators are fundamental in harmonic analysis since they serve as dyadic examples of Calderón-Zygmund operators. The boundedness of $\mathrm{P}_b^{\epsilon,\delta}$ is well-known and the result depends on the choice of ϵ, δ and the symbol b. We were interested in the question of the boundedness of the composition

$$\mathrm{P}_b^{\epsilon,\delta}\,\mathrm{P}_d^{\epsilon',\delta'} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

when the individual paraproducts are *not* necessarily bounded. In particular, we wanted to obtain a characterization of the boundedness of the above composition for all possible choices of $\epsilon, \epsilon', \delta, \delta' \in \{0,1\}$. Additionally, there is a formulation of the question that removes the function b and d, replacing them with sequence $\{b_I\}_{I\in\mathcal{D}}$ and $\{d\}_{I\in\mathcal{D}}$ and seeks a characterization in terms of sequence information on $\{b_I\}$ and $\{d_I\}$.

2 Scientific Progress Made

During the workshop we were able to make substantial progress on characterizing the various paraproduct compositions that exist.

2.1 The Simple Compositions

We now note that the composition $\mathsf{P}_b^{(\alpha,0)} \circ \mathsf{P}_d^{(0,\beta)}$ coincides with the paraproduct $\mathsf{P}_{b\circ d}^{(\alpha,\beta)}$ where $b \circ d$ is the Schur product

$$b \circ d \equiv \{b_I d_I\}_{I \in \mathcal{D}} \ .$$

Thus the boundedness of a composition operator of the form $\mathsf{P}_b^{(\alpha,0)} \circ \mathsf{P}_d^{(0,\beta)}$ reduces to that of a single paraproduct $\mathsf{P}_{b\circ d}^{(\alpha,\beta)}$. There are three cases in which a single paraproduct is easily characterized, namely $\mathsf{P}_a^{(0,0)}$, $\mathsf{P}_a^{(0,1)}$ and $\mathsf{P}_a^{(1,0)}$. Define

$$\begin{aligned} \|a\|_{\ell^\infty} &\equiv \sup_{I\in\mathcal{D}} |a_I| \, ; \\ \|a\|_{CM} &\equiv \sqrt{\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\subset I} a_J^2} \ . \end{aligned}$$

For these simple paraproducts we have the characterizations

$$\begin{aligned} \left\| \mathsf{P}_a^{(0,0)} \right\|_{L^2\to L^2} &= \|a\|_{\ell^\infty} \, ; \\ \left\| \mathsf{P}_a^{(0,1)} \right\|_{L^2\to L^2} &= \left\| \mathsf{P}_a^{(1,0)} \right\|_{L^2\to L^2} \approx \|a\|_{CM} \ . \end{aligned}$$

The next most difficult one is a paraproduct of type $(1,1)$. Using existing results from the literature, we were able to show that the single paraproduct $\mathsf{P}_a^{(1,1)}$ reduces to two types already characterized plus a diagonal operator $D_{(b,d)}$ that is easily characterized. Indeed, we have the norm of the bilinear form

$$\left\langle \mathsf{P}_a^{(1,1)} f, g \right\rangle = \sum_I a_I \langle f, A_I \rangle \langle g, A_I \rangle$$

on $L^2 \times L^2$ is comparable to

$$\left\| \sum_{I\in\mathcal{D}} a_I A_I \right\|_{BMO} + \sup_{I\in\mathcal{D}} \frac{1}{|I|} \left| \sum_{J\subset I} a_J \right| .$$

2.2 The Difficult Compositions

These cases presented a challenge because of the lack of cancellation present in the composition of two different paraproducts. The main idea we exploited was to rephrase the boundedness of the composition, as the boundedness of a related operator, but on a *different* Hilbert space. This translation then allowed the problem to be recast as a certain two weight inequality for an operator that could then be studied via the techniques of weighted theory from harmonic analysis.

The Hilbert space we found useful for studying the composition was the dyadic Bergman space

$$\mathcal{B} \equiv \left\{ f \in L^2(\mathcal{D}) : \sum_{I\in\mathcal{D}} |f(I)|^2 |I|^2 < \infty \right\},$$

with norm $\|f\|_{\mathcal{B}} = \sqrt{\sum_{I\in\mathcal{D}} |f(I)|^2 |I|^2}$. This can also be viewed as a weighted L^2 space on the tree $\mathcal{D}$.

2.2.1 Composing a Paraproduct of type $(0,1)$ with a Paraproduct of the $(1,0)$

Using the idea of recasting the problem into a different Hilbert space, we have shown that

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)} \right\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \approx \left\| \mathsf{T}_{b,d}^{(0,1,1,0)} \right\|_{\mathcal{B}\to\mathcal{B}} + \left\| \mathsf{T}_{d,b}^{(0,1,1,0)} \right\|_{\mathcal{B}\to\mathcal{B}}$$

with the operator $\mathsf{T}_{b,d}^{(0,1,1,0)}$ is an explicit operator acting on $\mathcal{B}$. Furthermore, the operator norm $\left\|\mathsf{T}_{b,d}^{(0,1,1,0)}\right\|_{\mathcal{B}\to\mathcal{B}}$ equals the best constant in a certain two weight inequality for the operator Q on $\mathcal{B}$, where

$$\mathsf{Q} \equiv \sum_{K\in\mathcal{D}} \widetilde{\mathbf{1}}_{T(K)} \otimes \widetilde{\mathbf{1}}_{Q(K)}.$$

Indeed, the inequality is

$$\mathsf{Q}\left(\widetilde{d}^2\cdot\right) : L^2\left(\widetilde{d}^2\right) \to L^2\left(b^2\right). \tag{1}$$

This best constant in this inequality is in turn comparable to the best constants in the associated testing conditions. These testing conditions thus give a characterization of the boundedness of the paraproduct composition $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$ on $L^2(\mathbb{R})$. Indeed, the two weight norm inequality (1) holds if and only if the following testing condition holds:

$$\begin{aligned}
\left\|\mathbf{1}_{Q_I}\mathsf{Q}\left(\sum_{J\in\mathcal{D}}\widetilde{d}^2\mathbf{1}_{Q_I}\right)\right\|_{L^2(b^2)}^2 &\lesssim \left\|\mathbf{1}_{Q_I}\right\|_{L^2\left(\widetilde{d}^2\right)}^2;\\
\text{i.e. } \sum_{J\subset I} b_J^2\frac{1}{|J|^2}\left(\sum_{L\subset J} d_L^2\right)^2 &\lesssim \sum_{L\subset J} d_L^2 .
\end{aligned}$$

Thus combining this the above discussion with the corresponding condition with b and d interchanged, we find that the composition $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$ is bounded on $L^2(\mathbb{R})$ if and only if both

$$\begin{aligned}
\sum_{J\subset I} b_J^2\frac{1}{|J|^2}\left(\sum_{L\subset J} d_L^2\right)^2 &\lesssim \sum_{L\subset J} d_L^2;\\
\sum_{J\subset I} d_J^2\frac{1}{|J|^2}\left(\sum_{L\subset J} b_L^2\right)^2 &\lesssim \sum_{L\subset J} b_L^2,
\end{aligned}$$

for all $I \in \mathcal{D}$.

2.2.2 Composing a Paraproduct of type $(0,1)$ with a Paraproduct of the $(0,0)$

Again, recasting the problem as an equivalent question on a discrete Bergman space, we have

$$\left\|\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}\right\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \approx \left\|\mathsf{T}_{b,d}^{(0,1,0,0)}\right\|_{\mathcal{B}\to\mathcal{B}} .$$

Now the operator norm $\left\|\mathsf{T}_{b,d}^{(0,1,0,0)}\right\|_{\mathcal{B}\to\mathcal{B}}$ equals the best constant in a certain two weight inequality for the operator $\widetilde{\mathsf{Q}}$ on $\mathcal{B}$ defined by

$$\widetilde{\mathsf{Q}} \equiv \sum_{K\in\mathcal{D}} \widetilde{\mathbf{1}}_{Q_\pm(K)} \otimes \widetilde{\mathbf{1}}_{T(K)}.$$

This operator is not positive, but its singular character is well-behaved, and the best constant in the two weight inequality is in turn comparable to the best constants in the associated testing conditions. These testing conditions thus give a characterization of the boundedness of the paraproduct composition $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$ on $L^2(\mathbb{R})$.

Using ideas for well-localized dyadic singular integral type operators we were able to show that

$$\widetilde{\mathsf{Q}}_\mu : L^2(\mu) \to L^2(\nu)$$

if and only if both

$$\begin{aligned}
\left\|\widetilde{\mathsf{Q}}_\mu\left(\mathbf{1}_{T(I)}\right)\right\|_{L^2(\nu)} &\lesssim \left\|\mathbf{1}_{T(I)}\right\|_{L^2(\mu)} = \sqrt{\mu(T(I))},\\
\left\|\widetilde{\mathsf{Q}}_\nu^*\left(\mathbf{1}_{Q(I)}\right)\right\|_{L^2(\mu)} &\lesssim \left\|\mathbf{1}_{Q(I)}\right\|_{L^2(\nu)} = \sqrt{\nu(Q(I))},
\end{aligned}$$

hold for all $I \in \mathcal{D}$. The weights μ and ν are given by explicit formulas in terms of the sequences b and d. These conditions can in turn be rephrased back in terms of testing conditions on the paraproducts in question.

2.2.3 The Remaining Cases

We still are working to finish up the remaining cases of the compositions that remain.

3 Outcome of the Research in Teams

We are currently writing up the results from the research in teams. We are optimistic that we will be able to resolve the final cases that remain.