Models for minimal Cantor $\mathbb{Z}^2$-systems

Thierry Giordano (University of Ottawa),
Ian F. Putnam (University of Victoria),
Christian F. Skau (Norwegian University of Science and Technology)

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1 Overview of the Field

The work of Herman, Putnam and Skau [6] used ideas from operator algebras to present a complete model for minimal actions of the group $\mathbb{Z}$ on a compact, totally disconnect metrizable space having no isolated points, i.e. a Cantor set. The data (a Bratteli diagram, with extra structure) is basically combinatorial and the two great features of the model were that it contained, in a reasonably accessible form, the orbit structure of the resulting dynamical system and also cohomological data provided either from the K-theory of the associated C^*-algebra or more directly from the dynamics via group cohomology. This led to a complete classification of such systems up to orbit equivalence [5]. This was the first extension of a famous program initiated by Henry Dye [2] in the study of orbit equivalence in ergodic theory to the topological situation. (See also [7, 1].)

The classification in [5] was extended to include minimal actions of $\mathbb{Z}^2$ in [3] and minimal actions of finitely generated abelian groups in [4]. However, what was not extended was the original model and this has handicapped the general understanding of these actions. The higher dimensional case is an important one, since, in particular, it has applications to the study of quasicrystals.

2 Recent Developments and Open Problems

A couple of years ago, the three of us had some new ideas on how to produce a model for actions of $\mathbb{Z}^2$. The two essential new insights were to start from the cohomological data, rather than the combinatorial data and secondly, to make some simplifying assumptions on the properties of the cohomology. The second point in particular is key; our knowledge of the general properties of the cohomology of minimal actions of $\mathbb{Z}^d$ is scant, when d exceeds one. (It is even conceivable that the problems involve some rather deep issues of decidability.) Of course, this means that the model can only produce dynamics whose cohomology satisfies these properties, but with the profound lack of success in discerning whether or not they hold in general, it seems a reasonable special case.

3 Scientific Progress Made

We have been able to establish most of the construction rigorously. During the stay at BIRS, we were able to show that the first cohomology group for the model was correct. We also made progress in computing the second. Rather remarkably, it seems that the second actually agrees with the data from (some) Bratteli diagram, which is more than we had expected. One of the crucial technical steps involves the structure of Bratteli diagrams having associated dimension groups with exactly two states. Here, we established some explicit estimates. Still remaining are some subtle estimates regarding the geometry of the finite approximations, but it seems unlikely that this problem is serious.

We also had plans to extend the model to admit more than one invariant measure and also to allow for torsion in the cohomology. On the former issue, initial investigations looked promising, but we obtained few hard results. No real progress was made on the latter.

4 Outcome of the Meeting

We made considerable progress in obtaining complete proofs of many of the technical estimates. We started writing these, which will become part of the final paper.

5 Anticipated Impact

We anticipate the result will have impact in a number of ways.

1. To provide a much broader class of examples of actions of $\mathbb{Z}^2$.

   While the model as we have it is not complete (it does not produce every minimal Cantor action of $\mathbb{Z}^2$), it should significantly extend the class of known examples, which is surprisingly small.

2. To clarify the structure of minimal actions of $\mathbb{Z}^2$.

   This should be an immediate consequence of having a larger class of examples and a systematic way of producing them.

3. Understanding the range of the cohomological invariants for $\mathbb{Z}^2$-actions

   While in the case of $\mathbb{Z}$, the range of the cohomology invariant is completely understood, this is still a complete mystery for higher rank actions. In particular, as a consequence of the orbit equivalence classification, it may be possible to have minimal actions of $\mathbb{Z}$ which are not orbit equivalent to any action of $\mathbb{Z}^2$. This situation does not occur in ergodic theory.

4. To emphasize the importance of the ring structure in cohomology.

   This ring structure is trivial for $\mathbb{Z}$-actions and while many computations of cohomology groups have been made in higher rank examples, the ring structure has mostly been ignored. It plays a crucial part in our model which should increase its profile in general.

5. To focus attention on the hypotheses used on the cohomology.

   As mentioned above, the structure of the cohomology groups for higher rank actions is not well-understood. Our construction makes some serious hypotheses on these groups as its starting point. It therefore draws attention to this lack of understanding and may encourage investigations into these issues.

References

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