Spectral and asymptotic stability of nonlinear Dirac equation

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1 Overview of the field and recent developments

The Dirac equation remains of utmost importance in High Energy Physics and Solid State Physics since its invention in 1928 [Dir28], and became the main building block of Quantum Electrodynamics. The idea to consider a self-interacting spinor field has appeared in Physics for a long time [Iva38, FLR51, FFK56, Hei57, Wak66].

The two most widely known models of self-interacting spinor fields are the massive Thirring model [Thi58], which is a model with the vector-vector (or *current-current*) self-interaction (in 1D, this model is completely integrable [Kor79]) and the model with the scalar-scalar self-interaction introduced by Soler [Sol70]:

$$i\frac{\partial}{\partial t}\psi = -i\sum_{j=1}^{n}\alpha_j\frac{\partial}{\partial x_j}\psi + m\beta\psi - (\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t)\in\mathbb{C}^4, \quad x\in\mathbb{R}^3,$$

where α_j and β are the 4×4 Dirac matrices. The one-dimensional version of the Soler model is also known as the massive Gross-Neveu model [GN74, LG75]. Let us also point out that the nonlinear Dirac equation is a representative of a wider class of models, known as *coupled mode equations*. Such equations appear in Solid State Physics [OKL$^+$00], in the theory of graphene [NGM$^+$05], and in Nonlinear Optics [dSS94, dSSS96, GWH01].

Dirac-type equations with the self-interaction of local type have been widely considered in Physics since the seventies. There were many results on the existence of localized solutions (*solitary waves*, or *gap solitons* of the form $\phi(x)e^{-i\omega t}$, with $\phi\in L^2(\mathbb{R}^n)$. The existence of solitary waves in the Dirac equation was already justified numerically in [Sol70]. For the Dirac-Maxwell system, the existence of solitary waves was indicated in [Wak66, Lis95]. The stability properties of these localized solutions, which are of crucial importance for understanding the role they could play (and also for applications e.g. in nonlinear optics), have remained a complete mystery, and were accessed either by numerics [RnRnSV74, AS83, AS86, BC12a, MQC$^+$12, XST12] or by heuristic arguments [Sol70, Bog79, MM86, SV86, BSV87, CKMS10]; essentially, no definite results were ever produced.

In the recent time, properties of the nonlinear Dirac equation and related models are being addressed by the mathematical community. Existence of solitary waves of the nonlinear Dirac equation was rigorously justified in [CV86, Mer88, ES95]; for the Dirac-Maxwell system, it was rigorously proved in [EGS96, Abe98].

Local and global well-posedness was addressed in [EV97] (semilinear equation), [Bou00] (interacting Dirac and Klein-Gordon equations in 1D), and in [MNNO05] (global well-posedness for the nonlinear Dirac equation in 3D, for small initial data in the energy class with additional regularity assumption for the angular variables). More recent results include [DFS10] (almost optimal well-posedness for Dirac-Maxwell) and [DS11] (global well-posedness for the nonlinear Dirac-Maxwell in 2D). Uniqueness of finite energy solutions to Dirac-Maxwell was studied in [MN03].

We are interested in developing the analytic approach to the stability properties of solitary wave solutions to the nonlinear Dirac equation and related systems. The pivotal papers by Zakharov [Zak67], Vakhitov and Kolokolov [VK73], Cazenave and Lions [CL82], Weinstein [Wei85], and Grillakis, Shatah, and Strauss [GSS87] gave the exhaustive analysis of spectral and orbital stability in the nonlinear Schrödinger equation and many similar $\mathbf{U}(1)$-invariant models; The nonlinear Dirac equation remained off-limits. Earlier heuristic approaches which we mentioned above were based on considering whether the energy is minimized or not under certain families of perturbations on the hypersurface of constant charge. These criteria were of doubtful validity, though; Let us point out that even the well-known semiformal instability result known as Derrick's theorem [Der64], which is based on the analysis of the values of the energy functional under dilations, while correctly predicting instability of stationary localized solutions in 3D (and in higher dimensions), fails to show such an instability in 1D and 2D. On the other hand, this instability easily follows by spectral theory methods [KS07], suggesting that the spectral stability approach is not only more rigorous mathematically, but also more precise.

In the last three years, the participants of the proposed focused research group made several breakthroughs which allow us to rigorously study the spectral stability and the asymptotic stability. We hope that combining our efforts will make the nonlinear Dirac equation as clearly accessible to stability analysis as the nonlinear Schrödinger equation and similar equations.

2 Presentation highlights

Here are some of the presentations which took place.

- Andrew Comech described the results on spectral instability on the nonlinear Dirac equation in 3D and higher-order nonlinear Dirac equation in 1D and 2D [CGG12].
- Kenji Nakanishi described his work on the Dirac-Maxwell system [MN03], focussing on the stability, spectral and nonlinear, of such a system. The related recent works of Piero d'Ancona on the wellposedness are [DFS05, DS11].
- Nabile Boussaid gave a presentation on a recent work [BC12b], discussing the recent progress on spectra stability of solitary waves in the Soler model.
- Tetsu Mizumachi described the related results in the context of the Kadomtsev-Petviashvili equation basing on his recent work [MT12].
- Slim Ibrahim gave a presentation on finite-time blow-up results for hyperbolic equations [IL12].
- Atanas Stefanov presented latest developments on asymptotic stability results for the nonlinear Dirac equation in the external potential [PS12].

The main conclusion is that the nonlinear Dirac equation and related systems have solitary wave solutions which are spectrally stable, and that the local and global well-posedness is known, at least to some extent. We expect that the problems on asymptotic stability of solitary waves in Dirac-Maxwell system are well within the reach of today's methods of nonlinear Analysis. This may be counter-intuitive to many researchers because the energy functional is unbounded from below, so that the energy conservation does not provide bounds which are routinely available to people who work on nonlinear Schrödinger and Klein-Gordon type equations.

3 Nonlinear Dirac equation and related models: Open problems

During discussions, the following promising directions of research were pointed out.

3.1 Maxwell-Dirac system (3D, also 2D, 1D)

1. in 3D, well-posedness in the critical space (L^2 for the spinor). The well-posedness in H^ϵ is due to d'Ancona et. al [DFS10].
2. In 2D, growth of H^1 norm (global well-posedness in L^2 is known)
3. Orbital and asymptotic stability of ground states coming from the nonrelativistic limit
4. Well-posedness/existence of ground states away from non-relativistic limit (eg. via Min-Max of [EGS96])
5. Orbital stability of such ground states
6. Instability of sign-changing standing waves (*excited states*) bifurcating from the nonrelativistic limit. This question is still open even for Schrödinger-Poisson system and for NLS.

3.2 Schrodinger-Poisson system (with indefinite Hamiltonian)

1. Non-degeneracy of sign-changing standing waves (to allow for the analysis of nonrelativistic-limit-type bifurcations of standing waves of Maxwell-Dirac)
2. Orbital/asymptotic stability of ground states $(0, Q_w)$
3. Related: dispersive estimates for linearization of the Hartree equation about ground states
4. Linear Schrodinger with $1/|x|$ potential

3.3 Dirac Models in 1D

1. L^2 ill-posedness in general ($H^{1/2}$ well-posedness is known; Thirring model is globally well-posed in L^2)
2. Well/ill-posedness in $H^{1/2}$
3. Asymptotic stability of ground states: cubic & quintic NLD; cubic NLS (cf. Cuccagna, via the approach of Deift); NLKG
4. Small-data scattering
5. Asymptotic stability of traveling waves of massless NLS (nonlinear transport)

3.4 Dirac Models in higher dimensions

1. In any dimension: NLD with potential supporting several eigenvalues: which corresponding nonlinear bound states are stable? unstable? Is it always possible to uniquely indicate the "ground state"?
2. Related question: instability of "nonlinear excited states" in NLS
3. 2D: stability/instability (spectral/orbital/asymptotic) of ground states for cubic NLD coming from the nonrelativistic limit, where stability is "marginal": the pair of purely imaginary eigenvalues is located asymptotically close to the non-degenerate zero eigenvalue.

3.5 Miscellaneous

1. Cubic Klein-Gordon: small-data scattering for nonlinearities below L^2-critical, down to "Strauss exponent"
2. (Non-)unconditional uniqueness for any of the equations above
3. To formulate the "Dirac Map" problem

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