Advancing numerical methods for viscosity solutions and applications (11w5086)

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Viscosity solutions are a well established analytical tool to treat highly nonlinear Partial Differential Equations. Following the emergence of an analytical theory, a number of numerical strategies have been studied, the interest in this field being both the development of efficient schemes and their use in real-world applications of the theory. The purpose of this workshop was precisely to collect together viscosity solution experts from applied mathematics, analysis and numerical methods as well as scientists and engineers researching related applications, so as to exchange and integrate ideas from these various fields.

The workshop has been co-sponsored by the research project and training network SADCO (see the homepage at `http://itn-sadco.inria.fr/`), which has provided travel funds for some participants, mainly doctoral and postdoc students. A further co-sponsorship has been provided by IMACS, who has agreed to publish a selection of papers from the workshop as a special issue of the journal Applied Numerical Mathematics.

The workshop has tried to give a reasonably complete and self-contained panorama of the current lines of research, also by means of some tutorial sessions, devoted to relevant recent advances in the field and especially directed to the younger portion of the audience.

1 Overview of the Field

The typical problems addressed in the theory of viscosity solutions are nonlinear, first- and second-order Partial Differential Equations of either the stationary form

$$H(x, u, Du, D^2u) = 0 \tag{1}$$

or the evolutive form

$$\begin{cases} u_t + H(x, u, Du, D^2u) = 0, \\ u(x, 0) = u_0(x) \end{cases} \tag{2}$$

(where Du and D^2u stand for respectively the gradient and the Hessian of u), with suitable boundary conditions. A certain number of problems of interest in applications may be put in the form (1) or (2), and among others:

- The Dynamic Programming (or *Bellman*) equation for the value function of finite or infinite horizon optimal control problems (see [1]) in which

$$H(x, u, Du, D^2u) = \lambda u + \sup_{\alpha} \left[-f(x, \alpha) \cdot Du - g(x, \alpha)\right]$$

Here, $\lambda \geq 0$, $f(x, \alpha)$ and $g(x, \alpha)$ represent respectively the controlled dynamics and the running cost, and the Hessian matrix D^2u may appear in the case of stochastic control problems. A generalization to the case of differential games (the so-called *Isaacs* equation) is also possible;

- The equation of the Shape-from-Shading (SfS) problem, in which $u(x)$ is the height of a lambertian surface which has irradiance $I(x)$. In the classical model for the SfS (see [5]), u solves the stationary equation

$$|Du(x)| = \sqrt{\frac{1 - I(x)}{I(x)}};$$

- The equation of front propagation in *level set* models (see [7, 10]), in which

$$H(x, u, Du, D^2u) = c(x)|Du|$$

where a front at time t is represented as a level set of $u(x, t)$, and $c(x)$ represents its speed of propagation and may contain terms related to curvature (hence, to D^2u) or to nonlocal properties. A classical example is provided by the choice

$$H(x, u, Du, D^2u) = -\mathrm{div}\left(\frac{Du}{|Du|}\right)|Du|$$

which corresponds to the level set model of the propagation by Mean Curvature.

The theory of viscosity solutions (see [1] for an extensive and self-contained treatment of the topic) is conceived to give a sound theoretical framework to equations like (1)–(2). Their analytical difficulties typically include strong nonlinearities, degeneracies, and the need for treating nonsmooth solutions in most relevant applications. In addition to such analytical complications, the application of viscosity solutions to the Dynamic Programming framework also requires working in a prohibitively high number of dimensions.

From a numerical viewpoint, the application of classical monotone schemes (upwind, Lax–Friedrichs) has been proposed from the very start of the theory. The first reference in this line is a work by Crandall and Lions [3], related to the numerical approximations of first-order HJ equations of the form

$$u_t + H(Du) = 0,$$

and inspired by the convergence theory of monotone schemes for conservation laws. This result has later been generalized in [2] to cover more general equations, including second-order equations like (1)–(2) with the only assumption of satisfying a comparison principle.

So far, the use of monotone schemes is supported by a stable and reliable theory, but has not proved efficient enough for the most challenging applications. On one hand, the low convergence rate requires a large number of nodes, and therefore a high computational complexity, as soon as a good accuracy is required. On the other hand, monotone schemes typically exhibit a highly viscous behaviour – this makes it difficult to locate the singularities of the solution, which are usually of interest in the applications (for example, they correspond to switching surfaces in dynamic programming equations). All such drawbacks are even more apparent when considering that some of the application we have in mind imply a high number of dimensions.

Hence, the search for more advanced and efficient techniques has become a key topic for applications. Among the main features of numerical techniques of interest for the field, we may list:

- a good cost-effectiveness, that is, a high convergence rate with respect to computational load;
- a good capability to approximate nonsmooth solutions, that is, a low numerical dispersion;
- the possibility to be implemented in high dimension with a reasonable complexity;
- a robust convergence theory.

We will examine in the sequel some of the techniques that have been proposed to answer these issues.

2 Recent Developments and Open Problems

Taking into account the recent advances in the field, along with the research lines represented in the workshop, the organizers have chosen to give a special emphasis to some specific direction of work, and more precisely: Dynamic Programming control techniques, Discontinuous Galerkin schemes, Semi-Lagrangian schemes and fast solvers (Fast Sweeping, Fast Marching). Four tutorials have been organized on such topics, and this has allowed in particular the youngest portion of participants to fill possible gaps. Moreover, looking at the contributed talks, these four subjects have proved to be real state-of-the-art work lines.

We briefly review the state of related researches.

Dynamic Programming

Dynamic Programming approach to optimal control problems (which characterizes a control problem via the associated Bellman equation, see [1]) dates back to the 1950s and has been widely studied in the applied mathematical community since, but has always been considered an impractical method because of the so-called *curse of dimensionality*, the typically exponential increase in complexity at the increase of the dimension.

However, while the real-life, massive optimal control problems remain out of the reach, it is clear that the continuous growth in the performance of computers has widened the range of applicability of Dynamic Programming techniques. In addition to this *brute force* increase, last years have witnessed the development of more efficient numerical schemes, in terms of high-order discretizations, fast solvers, sparse grids and so forth.

A tutorial on the relationships between Optimal Control problems and Hamilton–Jacobi equations has been given by Ian Mitchell, who has also recalled the concept of viscosity solution and discussed some application to real-world problems, although in low state-space dimension.

Discontinuous Galerkin schemes

Among the various high–order techniques, Discontinuous Galerkin (DG) schemes have originally been proposed in the framework of linear hyperbolic equations [8], and after a certain number of generalizations, they have been given a stronger theoretical framework in the last decade. The application of DG methods to Hamilton–Jacobi equations has been first proposed in [6] and seems to be a promising line of research. One one hand, they share the high geometric flexibility of finite element methods, and on the other they are well-suited for problems with nonsmooth solutions. At the moment, their main drawback is computational complexity.

A tutorial on DG methods has been given by Fengyan Li, a reputed expert in the field, who has reviewed the main concepts of DG schemes, as well as their construction in the case of first-order Hamilton–Jacobi equations.

Semi-Lagrangian schemes

Semi-Lagrangian (SL) schemes have a long tradition (set up in the 60s-70s) in the fields of plasma physics and Numerical Weather Prediction (see [11]), but have also proved to be effective in approximating viscosity solutions (see [4] and the references therein).

The main advantage of this class of schemes is their capability to work at large time steps, at the obvious price of a higher complexity for a single step of the scheme. On the other hand, large time steps also imply a low numerical viscosity – a key feature when working with nonsmooth solutions. SL schemes also admit a very natural formulation when approximating Dynamic Programming problems.

A tutorial on SL schemes has been given by Roberto Ferretti, who has shown the principles of construction, along with the basic convergence theory, in the case of linear advection equations and first-order HJ equations.

Fast solvers

Starting from the original versions of the Fast Marching Method (FMM) and the Fast Sweeping Method (FSM), first proposed respectively by J. Tsitsiklis in [13] and by Zhao, Osher, Merriman and Kang in [14] (see also the review paper [9]), the search for non-iterative solvers of static HJ equations has experienced a number of improvements and generalization in the last years, including nonhomogeneous, nonisotropic and nonmonotone propagation of the solution, as well as the use of unstructured meshes.

The tutorial on fast solvers has been split in two parts, respectively devoted to FM and to FS Methods, which have been given by Alexander Vladimirsky and by Hongkai Zhao. This has provided new material and discussion for the difficult, unsolved task of comparing the two techniques.

3 Presentation Highlights

We give an overview of the talks presented at the workshop.

- **M. Akian**, *Max–Plus algebra in the numerical solution of Hamilton–Jacobi and Isaacs Equations*

  A max-plus algebra approach to solve Hamilton–Jacobi and Isaacs equations is presented. First, the max-plus linearity of first order Hamilton–Jacobi equations allows one to construct approximations of their solutions by max-plus linear combinations, then use max-plus tools for their study. Second, monotone discretizations of stationary Isaacs equations yield to fixed point or spectral equations of dynamic programming operators of zero-sum, two player stochastic games.
  Using Perron–Frobenius properties inspired by the max-plus linear special case, a policy iteration algorithm has been introduced for degenerate (multi-chain) equations. The talk shows how policy algorithms behave combined with multigrid methods, and how this combination asks max-plus questions.

- **J.-D. Benamou**, F. Collino, S. Marmorat, *Local High Frequency wave content analysis*

  Given multi frequency domain wave data, the proposed new algorithm gives a pointwise estimate of the the number of rays, their slowness vectors and corresponding wavefront curvature. With time domain wave data and assuming the source wavelet is given, the method also estimates the traveltime. We present numerical results on synthetic data that demonstrate both the effectiveness and the robustness of the new method. Comparisons with more classical algorithms tends to show the superiority of the new method.

- Y. Cheng, **O. Bokanowski**, C.-W. Shu, *Discontinuous Galerkin scheme for front propagation with obstacle*

  The talk considers front propagation problems in the presence of obstacles, modeled by

  $$\min(u_t + H(x, \nabla u),\ u - g(x)) = 0,$$

  where u is a level set function and $g(x)$ is an obstacle function (Bokanowski, Forcadel and Zidani, SICON 2010). Following the lines of Cheng and Shu (JCP 2007), a direct Discontinuous Galerkin method for this Hamilton–Jacobi equation is proposed, for which in some special cases it is also possible to prove stability estimates for standard fully explicit RK DG schemes. Several numerical examples of front propagation are given to illustrate the efficiency of the method, and also the application of a narrow band approach is investigated.

- **F. Camilli**, D.Schieborn, *Shortest paths and Hamilton-Jacobi equations on a network*

  This talk presents an extension of the theory of viscosity solutions to topological networks. Uniqueness, existence and approximation results for Hamilton-Jacobi equations of eikonal type are discussed.
  A prominent question in graph theory is how to efficiently detect shortest paths connecting a given vertex with prescribed source vertices in a weighted graph. A similar problem is studied, assuming that the running cost varies in a continuous way along the edges.

- **E. Carlini**, *A Generalized Fast Marching Method on unstructured grids*

  Recently, a new version of the Fast Marching Method (called Generalized Fast Marching Method, GFMM) ha been proposed to treat the case of evolutive eikonal equations with speed that can change sign in time. In this talk, the extension of GFMM to unstructured grids is presented. The motivation for this work comes from several applications, in which methods to track interfaces are coupled with

other solvers (typically, finite elements or finite volumes) built on unstructured grids.
A general convergence result based on the properties of the local solver is given, and some numerical tests are presented.

- S. Cacace, **E. Cristiani**, M. Falcone, *Two new Ordered Upwind Methods for Hamilton–Jacobi equations*

  Two generalizations of the Fast Marching Method for the numerical solution of the eikonal equation are presented. The new methods are based on a semi-Lagrangian discretization and are suitable for Hamilton–Jacobi equations modeling monotonically advancing fronts, including anisotropic front propagation problems, Hamilton–Jacobi–Bellman and Hamilton–Jacobi–Isaacs equations. The algorithms are compared with classical Fast Marching and Fast Sweeping methods.

- **N. Forcadel**, *Generalized Fast Marching Method and applications*

  The Fast Marching Method have been proposed by Sethian in 1996 to solve very efficiently front propagation problem when the front evolves in its normal direction with a positive speed depending only on space.
  The goal of this presentation is to give a generalization of this algorithm when the normal velocity depends also on time and can change sign. It can be proved that the proposed algorithm is convergent and that the complexity is essentially the same as in the classical case.
  Finally, some applications in dislocations dynamics and image segmentation are presented.

- F. Cagnetti, **D. Gomes**, H. Tran, *Adjoint methods for obstacle problems and weakly coupled systems of PDE*

  In this talk, some new results for obstacle problems and weakly coupled systems of PDE are presented. The adjoint method, recently introduced by L. C. Evans, is used to study obstacle problems, weakly coupled systems, cell problems for weakly coupled systems of Hamilton–Jacobi equations, and weakly coupled systems of obstacle type. In particular, new results about the speed of convergence of common approximation procedures are derived.

- O. Bokanowski, J. Garcke, M. Griebel, **I. Klompmaker**, *A Semi-Lagrangian scheme using adaptive sparse grids for front propagation*

  Sparse grids are a technique for treating high-dimensional interpolation of functions without going through the typical exponential increase of complexity. This suggests that they could be a tool for applying SL schemes in higher dimensions.
  The talk reviews the basic facts about sparse grid approximations, and presents the application of sparse grids to the construction of a SL scheme for both linear advection and eikonal propagation of fronts. The performances of the scheme are evaluated on a set on numerical tests, and open theoretical issues are discussed.

- **J.-C. Nave**, *On some high-order, optimally local schemes for interface problems*

  The talk will present two schemes, one for the advection equation and the other for Poisson's equation with interface discontinuities. The peculiarity of these schemes is that they are local, but can achieve 4th order convergence (in the L^∞ norm). The basic idea hinges on the locality of Hermite basis, and that of the ghost fluid method. Some applications, as well as current thoughts on various extensions will be discussed.

- **A. Oberman**, *Numerical methods for geometric elliptic Partial Differential Equations*

  Geometric Partial Differential Equations can be used to describe, manipulate and construct shapes based on intrinsic geometric properties such as curvatures, volumes, and geodesic lengths. They have proven useful in applications (such as Image Registration and Computer Animation) which require geometric manipulation of surfaces and volumes. A few important geometric PDEs which can be solved using a numerical method called Wide Stencil finite difference schemes will be discussed: Monge-Ampere, Convex Envelope, Infinity Laplace, Mean Curvature, and others.
  Focusing in on the Monge-Ampere equation, it will be shown how naive schemes can work well for

smooth solutions, but break down in the singular case. A robust and efficient solver will be discussed, with a complexity comparable to solving the Laplace equation a few times.

- **S. Serna**, *Hamilton-Jacobi equations with shocks arising from general Fokker–Planck equations: analysis and numerical approximation*

  A class of Hamilton–Jacobi equations whose solutions admit shocks is considered. This class of equations arise as the convective part of a general Fokker–Planck equation ruled by a nonnegative diffusion function that depends on the unknown and on its gradient. The main features of the solution of the Hamilton–Jacobi equations are reviewed and a suitable numerical scheme is proposed in order to approximate the solution in a consistent way with respect to the solution of the associated Fokker–Planck equation. The talk also presents a set of numerical results performed under different piecewise constant initial data with compact support for specific equations including the one- and two-dimensional relativistic Hamilton–Jacobi equation and the porous media Hamilton–Jacobi equation.

- W. Chen, Z. Clawson, and S. Kirov, R. Takei, **A. Vladimirsky**, *Optimal control with budget constraints and resets*

  Many realistic control problems involve multiple criteria for optimality and/or integral constraints on allowable controls. This can be conveniently modeled by introducing a budget for each secondary criterion/constraint. An augmented HJB equation is then solved on an expanded state space, and its discontinuous viscosity solution yields the value function for the primary criterion/cost. This formulation was previously used by Kumar & Vladimirsky to build a fast (non-iterative) method for problems in which the resources/budgets are monotone decreasing.
  A more challenging case is addressed, in which the resources can be instantaneously renewed (& budgets can be ”reset”) upon entering a pre-specified subset of the state space. This leads to a hybrid control problem with more subtle causal properties of the value function. The problem is illustrated by finding (time-or-energy) optimal trajectories for a robot in a room with obstacles, constrained by the maximum contiguous time of visibility to a stationary enemy observer.

- S. Luo, **Y. Yu**, H. Zhao, *A new approximation for effective Hamiltonians for homogenization of a class of Hamilton–Jacobi equations*

  This talk presents a new formulation to compute effective Hamiltonians for homogenization of a class of Hamilton–Jacobi equations. The formulation utilizes an observation made by Barron and Jensen about viscosity supersolutions of Hamilton–Jacobi equations. The key idea is how to link the effective Hamiltonian to a suitable effective equation. The main advantage of such a formulation is that only one auxiliary equation needs to be solved in order to compute the effective Hamiltonian $\bar{H}(p)$ for all p. Error estimates and numerical examples are be presented.

- **H. Zidani**, *Convergence result of a non-monotone scheme for HJB equations*

  The convergence of a non-monotone scheme for one-dimensional first order Hamilton–Jacobi–Bellman equations of the form
  $$\begin{cases} v_t + \max_\alpha(f(x,\alpha)v_x) = 0 \\ v(0,x) = v_0(x) \end{cases}$$
  is presented. The scheme is based on the anti-diffusive method UltraBee. It is shown that, for general discontinuous initial data, a first-order convergence of the scheme towards the viscosity solution is achieved, in L^1-norm. The non-diffusive behavior of the scheme and its relevance are illustrated on several numerical examples.

4 Scientific Progress Made

Latest advances of numerical techniques

The organizers have made an effort to bring together the most up-to-date trends in the numerical analysis of viscosity solutions. The meeting has confirmed that development and theoretical analysis of efficient numerical schemes for viscosity solutions has had significant improvements in the last years, and in some cases (notably, Discontinuous Galerkin schemes and fast solvers) is likely to have further advances in the next future.

While most of the methodological work is carried out on academic test cases, a certain number of talks have presented numerical applications of less standard type and/or in higher dimension, which is encouraging for the application of advanced numerical techniques to real-world problems.

Interactions between various techniques

In order to address problems of interest for applications, it seems important that a careful mixing of different techniques could be implemented in practical computations. As a matter of fact, the interaction of various techniques has been exploited in many of the works presented. As an example, the application of fast methods has been studied in conjunction with different local solvers (monotone, Semi-Lagrangian, Discontinuous Galerkin), high dimensional techniques like sparse grids have been used in a Semi-Lagrangian framework, and so forth. This is an additional indication that the research is moving towards more challenging applications.

Key features and benchmark tests

A major obstacle to the use of viscosity solution approximation schemes in real-world problems – as in many parts of computational science – is that advanced algorithms are complex and hence difficult to implement, compare, and apply to specific problems. The organizers therefore held a discussion session on the final afternoon of the workshop to discuss the creation of a set of benchmark tests on which various algorithms could be implemented and compared. Such a suite of benchmarks can provide a variety of benefits to the community: a common basis for quantitative comparison of the strengths and weaknesses of specific approaches in publications, a common framework in which to teach and learn the variety of algorithms for newcomers to the field, a set of challenge problems important to application fields for which current schemes are unsuitable, etc. If algorithm designers are further willing to release their implementations of benchmark problems, then researchers from ! application fields will be able to adapt those algorithms to and test them on their problems much more easily. It is therefore hoped that a properly chosen set of benchmarks will both allow the field to grow more quickly and reach a broader audience. It has been agreed that the candidate benchmarks should contain some (or all) of the major difficulties encountered in this class of problems: inhomogeneous and/or anisotropic propagation of the solution, singularities and/or discontinuities in the solution, complex geometries, high dimensional setting, etc. It will also be useful for the candidates to have analytical solutions, or at least approximations determined by an alternative approach for validation and error analysis. Some benchmarks may be parameterized (in some way other than grid resolution) to permit studies of scaling behavior. The work has proved to be anything but trivial, but has raised great interest from participants and is still in progress.

5 Outcome of the Meeting

Summarizing the scientific products of the workshop, it is worth to point out that:

- The meeting has gathered 31 researchers in the field of numerical methods for viscosity solutions:

Akian, Marianne	INRIA Saclay and CMAP
Benamou, Jean-David	INRIA Roquencourt
Bokanowski, Olivier	Université Paris-Diderot
Cacace, Simone	Sapienza – Universitá di Roma
Camilli, Fabio	Sapienza – Universitá di Roma
Carlini, Elisabetta	Sapienza – Universitá di Roma
Cristiani, Emiliano	Sapienza – Universitá di Roma
da Silva, Jorge Estrela	Instituto Superior de Engenharia do Porto
Falcone, Maurizio	Sapienza – Universitá di Roma
Ferretti, Roberto	Universitá di Roma Tre
Festa, Adriano	Sapienza – Universitá di Roma
Forcadel, Nicolas	CEREMADE – Université Paris-Dauphine
Giorgi, Tiziana	New Mexico State University
Gomes, Diogo	Instituto Superior Tecnico Lisboa
Guigue, Alexis	University of British Columbia
Kao, Chiuyen	Ohio State University
Klompmaker, Irene	TU Berlin
Li, Fengyan	Rensselaer Polytechnic Institute
Mecca, Roberto	Sapienza – Universitá di Roma
Mitchell, Ian	University of British Columbia
Nave, Jean-Christophe	McGill University
Oberman, Adam	Simon Fraser University
Rao, Zhiping	ENSTA ParisTech
Serna, Susana	Universitat Autonoma de Barcelona
Sethian, James	University of California, Berkeley
Smits, Robert	New Mexico State University
Tasso de Figueiredo Borges de Sousa, Joao	Universidade do Porto
Vladimirsky, Alexander	Cornell University
Yu, Yifeng	University of California, Irvine
Zhao, Hongkai	University of California, Irvine
Zidani, Hasnaa	ENSTA ParisTech and INRIA Saclay

including young doctoral and postdoc students.

- Four tutorials have been devoted to the main research lines represented at the workshop; this has provided a good introduction for the younger participants, as well as a chance to compare the various techniques.
- The talks have shown less and less academic numerical examples, along with an increasing trend to mix the various techniques in order to approach the most challenging applications.
- A round table has been allocated to a discussion about creating an established set of benchmark problems in the field of viscosity solutions, in a spirit similar to what has been done in Computational Fluid Dynamics. This is an ongoing work (Mitchell).
- Up to the writers' knowledge, at least a couple of other ongoing researches have been started among participants of the workshop, in particular one concerning dynamic programming on hybrid control problems (Ferretti, Zidani), and the other on Fast Marching Methods (Cacace, Carlini, Cristiani, Falcone, Vladimirsky).
- Due to the co-sponsorship that IMACS has given to this workshop, it is expected that some of the works presented at the workshop will be collected in a special issue of the journal Applied Numerical Mathematics.

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