BIRS Workshop 11w5033
Linear Algebraic Techniques in Combinatorics & Graph Theory

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1 Overview

Linear Algebra and Matrix Theory provide one of the most important tools–sometimes the only tool–in Combinatorics and Graph Theory. Even though the ideas used in applications of linear algebra to combinatorics may be very simple, the results obtained can be very strong and surprising. A famous instance is the Graham-Pollak theorem which asserts that if the complete graph of order n is partitioned into m complete bipartite subgraphs, then m is at least n-1 (n-1 arises naturally by recursively deleting a star at a vertex, but there are many other ways to achieve n-1). The only known proofs of this theorem use some form of linear algebra. How does linear algebra enter into this and other combinatorial problems?

Almost all combinatorial objects can be described by incidence matrices (e.g. combinatorial designs) or adjacency matrices (e.g. graphs and digraphs). Sometimes the Laplacian and Seidel matrices are also used. Therefore one basic approach is to investigate combinatorial objects by using linear algebraic parameters (rank, determinant, spectrum, etc.) of their corresponding matrices. Two of the great pioneers of this approach were H.J. Ryser and D.R. Fulkerson. Strong characterization and non-existence results can be obtained in this way. The application of linear algebra to combinatorics works in the reverse order as well. Many linear algebraic issues can be refined using combinatorial or graph-theoretic ideas. A classical instance of this is contained in the Perron-Frobenius theory of nonnegative matrices where use of an associated digraph gives more detailed information on the spectrum of the matrix. Another more recent, but now classical, instance is the use of an associated digraph to refine the classical eigenvalue inclusion region of Gershgorin.

Another classical instance of the use of linear algebra to get combinatorial information is the theorem of Bruck and Ryser which rules out the existence of finite projective planes of certain orders. A further instance is the Friendship Thoerem which states that a graph in which every pair of vertices have exactly one common neighbor is a bunch of triangles, glued together in one single vertex. The important class of highly structured graphs known as strongly regular (connected) graphs have a linear algebraic characterization: they are the graphs whose adjacency matrix have exactly three distinct eigenvalues. The Graham-Pollak theorem

mentioned above can be further generalized to graphs in general giving a bound on the integer parameter m in terms of the spectrum of the adjacency matrix.

Sometimes the lines between linear algebra and combinatorics are blurred. A simple example is the Cayley-Hamilton theorem that asserts that a matrix satisfies its characteristic polynomial. This theorem can be formulated and proved as a theorem in graph theory. Sign-nonsingularity of a matrix, where nonsingularity depends only on the signs $(+, -, 0)$ of its entries and not on its magnitudes, and the resulting theory and application to linear systems, is a both a linear algebraic issue and a combinatorial issue. The existence of Hadamard matrices of all orders $4m$ can be viewed as both a linear algebraic problem and a combinatorial design problem. Related to this basic combinatorial problem are bounds for the determinant of matrices of 1s and -1s. There are many more examples that can be given.

It seems to be the case and it has been conjectured that almost all graphs are determined by their spectrum (the adjacency spectrum as well as the Laplacian spectrum). This would mean that the spectrum can be used as a kind of fingerprint for a graph. Especially for large networks this is an interesting property which, for example, makes it possible to order these networks (in almost all cases) in a systematic way. In another direction it is shown that the null space of the incidence matrices of directed graphs, undirected graphs, and inclusion matrices of designs have correspondence with flows in graphs, zero-sum flows, and trades in Latin squares, respectively.

2 Objectives of the Workshop

The main objective in organizing this workshop was to bring together a broad representation of the large and diverse collection of researchers who have made substantial contributions using linear algebra techniques in combinatorics and graph theory, or using combinatorial and graph-theoretic ideas to investigate matrices. The expectation was that there would be considerable cross-fertilization of ideas leading to people learning new problems and new applications of linear algebra techniques, and that this would lead to new collaborations, insights, and breakthroughs. Recent successes in collaborations of this kind encouraged us in the belief that such a meeting could bring new perspectives on old problems and open up many channels of communication for future collaborative research. For young researchers in particular, it would be an excellent opportunity to see the wide scope of the subject and the many interesting directions that can be explored.

3 Participants

We feel that we were quite successful in enticing a strong and diverse group of participants from many different parts of the world. In addition to the organizers, the participants were:

Mahmud Akelbek (USA),
Sejong Bang (S.Korea),
Aart Blokhuis (The Netherlands),
Steve Butler (USA),
Bart De Bruyn (Belgium),
Charles Colbourn (USA),
Rob Craigen (Canada),
Carlos Fonseca (Portugal),
Shmuel Friedland (USA),
Wolfgang Holzmann (Canada),
Steve Kirkland (Ireland),
Reza Khosrovshahi (Iran),
Jack Koolen (S. Korea),
Christian Krattenthaler (Austria),
Esther Lamken (USA),
Felix Lazebnik (USA),
William Martin (USA),
Seth Meyer (USA),

Margarida Mitjana (Spain),
Ali Mohammadian (Iran),
Bojan Mohar (Canada),
Vladimir Nikiforov (USA),
Juan Rada (Venezuela),
Michael Schroeder (USA),
Bryan Shader (USA),
Azhvan Sheikh Ahmady (Canada),
Doug Stinson (Canada),
Hajime Tanaka (Japan),
Paul Terwilliger (USA),
Wei Wang (China),
Steven Wang (Canada),
Ian Wanless (Australia),
Richard Wilson (USA),
Yaokun Wu (China),
Mieko Yamada (Japan),

4 Program

Many participants were very eager to speak in order that the other participants would learn of their recent work and have the opportunity to react to it and provide additional connections and references. As a result, we had a full program of talks. In spite of the full program, there was ample time at coffee breaks, lunches, dinners, and evenings for informal discussions and collaborative work. As usual Wednesday afternoon was free for enjoying Banff and its surroundings, and on Friday morning there was a lively problem session. On Tuesday night, about one-quarter of the workshop participants enjoyed the Karaoke night at one of the local Banff establishments.

The full program, as well as the problems posed in the problem session, are available on the BIRS website, so we only provide here a summary of the topics of the talks. Very broadly speaking, the talks were on the following topics, all connected in one way or another wih linear algebra and matrix theory.

1. distance-regular graphs and other combinatorial configurations
2. Hadamard matrices
3. hyperplanes in finite geometries
4. spectral graph theory
5. graph matching
6. permanents, determinants and pfaffians in enumerative combinatorics
7. testing a database for certain characteristics
8. nowhere zero flows in graphs
9. extremal graph theory
10. permutation arrays
11. difference sets, orthogonal arrays, and codes
12. Smith normal form and hypergraphs

5 Participant Testimonials

The most effective way to convey the great benefit that participants received from the workshop is through their own testimonials. Here we give a sample of what some of the workshop participants reported after the workshop.

- My work falls entirely within the domain of what I describe as "Combinatorial Linear Algebra". It is a rare opportunity for me to attend a meeting that so consciously pays attention to this most fertile intersection of fields. The collection of experts assembled at BIRS for this event was very impressive, and I was able to initiate a number of very helpful conversations regarding various lines of research, and I was afforded an opportunity to explain a new and exciting type of construction for certain matrices, which I and my collaborators have recently developed, to an audience uniquely equipped to understand its significance. This kind of meeting provides valuable fertilization for my work, of a sort that is not easily found elsewhere, not because I do not occasional cross paths with these same people, but because when I do it is often not in a context so focussed on this kind of exploration, and generally not in such a broad group of people attuned to studying this kind of problem. (Robert Craigen)

- It was a great meeting, with great people, in a singular place. BIRS contains excellent conditions for research in Mathematics. I had the possibility to update some recent results on my field(s) of interest and get several ideas for a future research. (Carlos Fonseca)

- I found Craigen's talk on Buston-Hadamard matrices offered an interesting avenue for further investigation. The fair number of talks on Laplacian matrices and energy of graphs broadened my understanding of this area and has perked my interest in it. Finally the problem session gave a number of leads for problems which may be useful to give to graduate students to work on. (Wolfgang Holzmann)

- I received some helpful comments after my talk, including three different people who provided some references to related work – this sort of feedback will be quite valuable to me, as the research that I presented is a new area for me. Also, a new collaboration may emerge between myself and Juan Rada. After seeing his talk on energy for digraphs, I had some ideas as to how the problem might be addressed for tournaments, and I will correspond with Juan in the near future on that topic. (Steve Kirkland)

- I met several new people at this workshop. The talk of M. Mitjana made me realize that some of the research my student is doing is also useful for other people. For me the good point of this conference was that I met some old friends and the discussions I had with them really helped me to see where my next step in my research should be. Although this has no direct impact in the sense of a new paper or so, it was really helpful for me. (Jack Koolen)

- I enjoyed the Workshop a lot. I cannot claim that it led to new collaborations or new results. It was very instructive to see for me what is going on in fields that I am not so acquainted with. I was very glad to meet for the first time people (such as for instance Charles Colbourn, Felix Lazebnik, Ian Wanless) whose name I had known for long but had never met before. In the talks and in discussions I saw many inspiring problems. As I say, I cannot claim to be able to say something on them now, but I hope to do so at a later point. (Christian Krattenthaler)

- I had several very useful contacts, and they will result, hopefully, in future collaboration. Two participants decided to visit the University of Delaware this Spring to continue our discussions at the conference: W. Haemers (about chromatic number of Erdos-Renyi graph) and B. Shader (some linear hypergraph problems). I plan to continue some work I began with Vlado Nikiforov (on existence of special book-like configurations in graphs), and with Steven Wang concerning my conjecture on some class of permutation polynomials. It was very useful to meet with S. Akbari and other members of the group from Iran. Their research concerning the Alon-Tarsi conjecture was very interesting to me. We found several other common mathematical interests, exchanged some problems, and plan to continue communication. (Felix Lazebnik)

- More specifically, the discussions with some participants have given me new ideas that, I hope, will help to tackle problems that we have been dealing and we will possibly make progress in many of them.

Moreover, I have learn very recent results that I shall take into account in order to obtain new results. Finally, the last aspect I would like to emphasize is the benefit that represents to explain the own recent work to a such specialized audience and to collect their impressions. (Margarida Mitjana)

- Paul Terwilliger and I started work and made significant progress on the problem of characterizing the sign patterns of nilpotent tridiagonal matrices. (Bryan Shader)

- Thanks very much for organizing such a wonderful workshop. Indeed, I have made many new contacts and are interested in several problems presented in the workshop. The talks are diverse, but with the common practice of linear algebraic techniques. The length of talks are nice so that necessary backgrounds can be explained to general public. In particular, I become more interested in looking at some conjectures related to matrices over finite fields, which were discussed during the workshop. (Steven Wang)

- The workshop was a thoroughly worthwhile week for me. It allowed me to meet a number of big names in the field that had only ever been names on paper up to that point. In particular I was delighted to meet Friedland, who has done some of the most impressive work on permanents in the last few decades, and whose work has heavily influenced my own. I had some very interesting discussions and I learnt some useful results in the talk. To cap it all off, I was able to solve a problem that was posed on the first day (about existence of Latin retransmission permutation arrays). In time this will lead to a publication for me; and I believe it was also pleasing to Doug Stinson who posed the problem. Of course, the setting was just amazing. BIRS must be one of the most inspiring venues in the world (I can't think of anywhere that beats it). (Ian Wanless)

- I'd like to mention a few possible new collaborations after BIRS 11w5033. I talked with Professor Friedland about the graph isomorphsim problems and spectral characterization of graphs. He pointed out to me a paper of His in 1989 using Coherent Algebra for the purpose of graph isomorphism. So it's possible for us to make a collaboration to further study these relationship (Graph Isomorphsim, Spectral Characterization of Graphs and Cohentent Algebra); I discussed with Dr. Butler about the construction of graphs with the same generalized characteristic polynomial, which is not arised from GM-switching method. This could give some further resluts about my conjecture presented in the talk. I learned that Professor Steven Wang is an expert in Finite Field. Previously I have made some conjectures about the irreducibility of rational polynomials over rational fields. We decided to further communicate with each other to study these conjectures. (Wei Wang)

- I met Mieko Yamada from Japan the first time. I had read some of her papers on Hadamard matrices many years ago. So it was great to finally meet her and talk to her. Aart Blokhuis reminded us the Alon-Tarsi conjecture on even/odd Latin squares, and surveyed recent progress on the conjecture. Robert Craigen talked about his joint work with B. Compton and W. de Launney on constructions of Hadamard matrices, which are very interesting to me. (Qing Xiang)

- Prof. Xiang and I talked about the skew-Hadamard difference sets over elementary abelian groups. We also discussed about the Gauss sums and Jacobi sums over finite fields and Galois rings, and affirmed the importance of these concepts. We will discuss the applications of Gauss sums to difference sets and cyclotomy continuously after the workshop. I believe we could start new research. Prof. Craigen and I believed we could collaborate on the research of the generalized Hadamard matrices. The talk by Prof. Terwilliger gave the ideas to my research on association schemes over Galois rings. (Mieko Yamada)

- I enjoyed many wonderful talks there, either related to my research interest or out of my former research range. Especially, I learned from the workshop some progress on Erdos-Ko-Rado and on Alon-Tarsi, which are of much interest to some of my colleagues in Shanghai (Jun Wang, Jiayu Shao, Jiyou Li, etc.) and outside of Shanghai (Huajun Zhang, John Goldwasser), and I will surely forward what I learn here to them. (Yaokun Wu)

6 Conclusion

In our opinion, and as confirmed by the testimonials of participants, the BIRS Workshop 11w5033 was a tremendous success. It was a stimulating week for participants with a substantial amount of information exchanged, and new contacts and collaborations made. A week spent at a BIRS workshop can have a large effect on the research of participants. It can lead to new perspectives and new ideas on one's research, and can lead to avenues of research that were not anticipated before. We believe that our workshop illustrates this very clearly. There is nothing in the world like BIRS. It's unique setting and organization is having a profound effect on the development of mathematics. We hope that we might organize another similar BIRS workshop in the not too distant future.

The following references are mainly confined to books and survey papers in the general area of the workshop.

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