Triangulated categories and applications

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1 Overview of the Field

The best way to think of triangulated categories is as a powerful technical tool which has had amazing applications over the last 30 years. Homological algebra was first introduced quite early in the 20th century, and by the 1950s the subject was viewed as fully developed: so much so that the authoritative book, by Cartan and Eilenberg, dates from this decade, and until recently no one bothered to write an update. Around 1960 Grothendieck and Verdier introduced a new twist (triangulated categories), a refinement no one expected. It took about 20 years before people started realizing the power of the method, and in the last 30 years the applications have reached corners of mathematics no one would have foreseen. In organizing the conference we tried to bring in people from diverse fields, who use triangulated categories in their work, to give them a chance to interact. Many of the participants told us it was the best conference they had been to in a long lime.

2 Recent Developments and Open Problems

It's hard to know where to begin, there has been so much recent progress and it's been all over the place. Some of the progress has been on foundational questions, such as Brown representability [27, 11, 26, 5] and understanding the Balmer spectrum [1, 2, 4] of a tensor triangulated category. There has been considerable progress in understanding t-structures on various derived categories, for example the theory of Bridgeland stability conditions [8, 9, 10]. There are several theorems about obstructions to finding model structures for triangulated categories, examples of categories without model structures [25], and theorems telling us about the existence and uniqueness of model structures for large classes of triangulated categories [28, 23]. There has been major progress in computing the Balmer spectrum [3], as well as generalizations to categories with coproducts and applications in the specific cases [6, 7]. There has also been progress on applying triangulated categories to mathematical physics [12, 20, 21, 22] and to the study of C^*–algebras [24].

Every bit of progress opens up a myriad new questions, the very fact that we are making so much progress is evidence of how little we know.

3 Presentation Highlights

We began the conference with three very introductory presentations: we decided that, for such a diverse audience, there was a need to have experts present an overview of how their field finds triangulated categories relevant. Somewhat arbitrarily we decided to split up the applications into three broad areas: Mark Hovey talked about topology, Yujiro Kawamata introduced algebraic geometry, and Henning Krause spoke about representation theory. Of course this leaves out many of the fields to which triangulated categories are applied, but we couldn't very well have a conference that was just a string of introductory talks.

In choosing the speakers we kept diversity in mind; we tried to have as many fields as possible represented, and specifically asked each speaker to bear in mind that he or she will be addressing an audience most of whom know nothing about their specialty. As already mentioned, the participants were enthusiastic about the outcome.

Ralf Meyer and Ivo Dell'Ambrogio gave talks with a C^*–algebra perspective, Gonçalo Tabuada introduced us to how to reframe a whole bunch of conjectures in terms of suitable model liftings of triangulated categories and their localizations. Neil Strickland talked about the right axiomatization of tensor products in triangulated categories. Greg Stevenson and Srikanth Iyengar both gave talks about two (quite different) generalizations of the Balmer spectrum to infinite triangulated categories, that is to ones with coproducts, and each gave examples of applications. Jon Carlson and Julia Pevtsova both talked about applications to representation theory and group cohomology. Sabin Cautis and Tony Licata talked about categorified representation theory and the triangulated categories that arise there. John Greenlees told us about cohomological descriptions of complete intersection, with a derived category tilt. Brooke Shipley presented results on the equivariant version, where the group G is a compact Lie group; this improves on work by Greenlees in the special cases where G is either a finite group or the group S^1. Don Stanley gave a talk about the progress made in the attempts to classify t–structures in certain interesting cases. Fernando Muro told us about the current confused status of our understanding of Brown representability, including the (published) theorem of Rosický that unfortunately happens to be false and counterexamples to it. Wendy Lowen and Amnon Yekutieli each gave a talk about (totally different) results of interest in non-commutative algebraic geometry. And finally Daniel Murfet explained to us the problem of constructing finite dimensional models of the pushforward of a matrix factorization: it's easy to show, by theoretical arguments, that such a model exists, but an effective procedure for constructing one is very new.

It goes without saying that there were many topics we could not cover, and the fact that some topics were left out was because speakers chosen to cover them cancelled their participation at the last minute.

4 Scientific Outcomes of the Meeting

The meeting aimed to bring together a very diverse group of mathematicians who employ the methods of triangulated categories in totally different fields. They certainly were interested in listening to each other, and we hope that the interactions and exchange of ideas will continue. Over time we will be able to form a more accurate picture of what the scientific benefits will be.

5 Abstracts

Mark Hovey

To open this conference on triangulated categories and applications, we first describe how triangulated categories arose in algebraic topology. Thus we explain the process that leads from topological spaces to specta to the stable homotopy category. The stable homotopy category is a compactly generated tensor triangulated category. The most basic feature of any such category is the graded commutative ring of self-maps of the unit, which in this case consists of the stable homotopy groups of the sphere. We describe what is known about this ring, but it is so complex that there are no prospects for ever computing it entirely. So we need new approaches. We describe two such approaches, both worked out by Mike Hopkins and Jeff Smith. We first describe all the field objects of the category; these turn out to be the Morava K-theories. We then describe the thick subcategories of compact objects and the associated prime ideal spectrum, in the sense of Paul Balmer. Finally, we describe how the stable homotopy category gives rise to all of the commonly studied triangulated categories, including ones seemingly far away from algebraic topology. Building on work of Stefan Schwede and Brooke Shipley, Andreas Heider has recently shown that every well-generated triangulated category with a model is a localization of the triangulated category of modules over a symmetric ringoid spectrum.

Yujiro Kawamata

This was a survey talk on the structure of the bounded derived categories of coherent sheaves on algebraic varieties $D^b(X) = D^b(\mathrm{Coh}(X))$, viewed as new kind of spaces. I started with results on explicit structures given by exceptional collections of objects such as

$$D^b(\mathbf{P}^n) \cong \langle \mathcal{O}(-n), \ldots, \mathcal{O}(-1), \mathcal{O} \rangle$$

by Beilinson, and by semi-orthogonal decompositions such as

$$D^b(X) \cong \langle U^+, \mathcal{O}_X, D^b(C) \rangle$$

for a Fano 3-fold X of degree 12 and a curve C of genus 8 by Kuznetsov.

Then I went on to more complicated derived categories when the canonical divisor is trivial. I explained an equivalence called the Fourier-Mukai transform

$$\Phi_P : D^b(A) \to D^b(\hat{A})$$

for an abelian variety A and its dual abelian variety $\hat{A}$ defined as an integral functor whose kernel is the Poincaré line bundle. I also explained a more complicated case of K3 surfaces and the derived Torelli theorem.

I explained the representability theorem by Bondal and Van den Bergh, which says that the category $D^b(X)$ is saturated. I also explained the representability theorem by Orlov, which says that equivalences of derived categories are expressed as integral functors. I also talked about the Fourier-Mukai partners.

I talked about invariants of the categories. I explained Hochshild (co)homology, and the Hochshild-Kostant-Rosenberg isomorphism. I explained the moduli spaces of stability conditions defined by Bridgeland, and his structure theorem on the autoequivalence group of the derived category of a K3 surface.

Finally I explained the relationship between the semiorthogonal decompositions and the minimal model program. Especially I explained so-called K to D conjecture. I explained a problem of replacing the categories when the varieties have singularities.

Henning Krause

When one studies the representations of a finite dimensional algebra A (over some field k), there are two triangulated categories which naturally arise: the bounded derived category $D^b(\mathrm{mod}\, A)$ of the category $\mathrm{mod}\, A$ of finite dimensional representations, and the stable category $\mathrm{stmod}\, A$ where one takes the quotient modulo all morphisms factoring through a projective module. For the second construction, one needs to assume that the algebra is self-injective; a typical example would be the group algebra kG of a finite group G. An important example for the first construction arises from the path algebra kQ of a finite quiver Q without oriented cycles.

Historically, the first interesting results are those of Beilinson and Bernstein–Gelfand–Gelfand (both published in 1978). They provide a beautiful connection between algebraic geometry and representation theory and contain already a number of important ideas, which led in the following years to further exciting results. Beilinson describes for the projective space $\mathbb{P}^n_k$ an equivalence of the form $R\,\mathrm{Hom}(T, -)$ between $D^b(\mathrm{coh}\,\mathbb{P}^n)$ and $D^b(\mathrm{mod}\,\mathrm{End}(T))$ via a tilting object T, while the BGG-correspondence gives an equivalence between $D^b(\mathrm{coh}\,\mathbb{P}^n)$ and the stable category of graded modules over the exterior algebra of an $n+1$ dimensional vector space.

Then I gave the basic results from tilting theory, in particular Rickard's theorem characterising the fact that two algebras have equivalent derived categories. Also the stable category was explained in further detail, based on work of Buchweitz for Gorenstein algebras.

Next I moved to hereditary categories such that its derived category admits a tilting object. Happel's theorem tells us that there are just two types: the representations of path algebras and the coherent sheaves on weighted projective lines in the sense of Geigle–Lenzing.

Auslander-Reiten theory for triangulated categories and Serre duality form another interesting aspect. A theorem of Happel says that $D^b(\mathrm{mod}\, A)$ has AR-triangles if andf only if A has finite global dimension.

One of the great success stories in representation theory is based on the notion of cohomological support for modular group representations. A theorem of Benson, Carlson, and Rickard gives a complete classification of all thick subcategories of $\mathrm{stmod}\, kG$, using the support defined in terms of the group cohomology $H^*(G, k)$.

Finally, I moved to the theory of exceptional sequences which is well developed for quiver representations. A recent result of Ingalls and Thomas implies a classification of all admissable thick subcategories of $D^b(\operatorname{mod} kQ)$ in terms of non-crossing partitions associated to the underlying diagram of the quiver Q. Taking as an example the Kronecker quiver (two vertices and two parallel arrows), this classification complements the classification of the thick tensor ideals which one gets from the equivalence with $D^b(\operatorname{coh} \mathbb{P}^1_k)$.

Ralf Meyer

Noncommutative topology views C*-algebras as a generalization of pointed compact topological spaces. The basis of this is that the category of commutative C*-algebras is anti-equivalent to the category of pointed compact spaces by Gelfand's Theorem.

An important source of C*-algebras are stacks, which may be described by groupoids and corresponding C*-algebras. The C*-algebra associated to a groupoid is, however, only well-defined up to an appropriate C*-algebraic notion of Morita equivalence. Thus we add a Morita invariance condition to the axioms for homology theories in the C*-algebraic context.

This has rather drastic effects: it turns out that any Morita invariant and split-exact functor on C*-algebras behaves essentially like K-theory. In particular, the universal such functor deserves to be called bivariant K-theory. This is the analogue of the stable homotopy category in noncommutative topology. It is a triangulated category as well.

Several results in homotopy theory can be extended to noncommutative topology by first generalizing them to triangulated categries and then specializing to bivariant K-theory. In many cases, this requires some further work because C*-algebras do not admit uncountable coproducts or internal Hom functors. Hence standard notions from homotopy theory do not apply directly. It seems also quite difficult to relate bivariant K-theory to model categories.

One of the most powerful techniques in this context is homological algebra in triangulated categories. It provides an abelian approximation to a triangulated category, starting with a stable homological ideal. This is a powerful substitute for t-structures. The latter do not apply to bivariant K-theory because of Bott periodicity.

Approximating bivariant K-theory and its variants by Abelian categories also leads to some interesting phenomena in ring theory. The search for universal coefficient theorems for equivariant versions of bivariant K-theory has led to several examples of hereditary exact categories that deserve further study by algebraists.

Goncalo Tabuada

I will start by introducing the category $\mathcal{M}ot$ of *noncommutative motives*, as invisioned by Drinfeld and Kontsevich in their noncommutative algebraic geometry program. This category is the natural setting for the study of several classical invariants such as cyclic homology (and its variants), algebraic K-theory, topological Hochschild homology, etc. Among other results, I will show that algebraic K-theory becomes co-representable in $\mathcal{M}ot$ by the tensor unit object. Then, I will present joint work with Ivo Dell'Ambrogio on the tensor triangular geometry of noncommutative motives. Making use of Hochschild homology, periodic cyclic homology, and algebraic K-theory, I will describe some explicit points in the Balmer spectrum of $\mathcal{M}ot$. This will illustrate the complexity of the Balmer spectrum of $\mathcal{M}ot$. Finally, I will present joint work with Paul Balmer on the assembly isomorphism conjectures. Our main result asserts that the *fundamental assembly isomorphism conjecture* (which implies all the conjectures on the market) is simply a coefficients variant of the classical Farrell-Jones conjecture in K-theory. This will illustrate the central role played by the classical Farrell-Jones conjecture among all isomorphic conjectures.

Neil Strickland

Many important examples of triangulated categories come equipped with a symmetric monoidal tensor product. There are some basic features of the interaction between these structures that are well-known. However, Peter May has pointed out that there are more subtle interactions present in the usual examples, which are important for certain applications. Here we report on a project to make May's results more complete and symmetrical. For simplicity, we restrict attention to the homotopy category $\mathcal{D}$ of ungraded differential modules over a fixed commutative $\mathbb{Z}/2$-algebra. This has a canonical triangulation for which the suspension functor is the identity. We believe that, with more bookkeeping, our constructions can be generalised to cover all the usual algebraic examples of tensor triangulated categories.

Suppose we have two distinguished triangles in $\mathcal{D}$, say $A_0 \to A_1 \to A_2 \to A_0$ and $B_0 \to B_1 \to B_2 \to B_0$. By tensoring these together, we obtain a functor $F\colon \mathcal{U} \to \mathcal{D}$, where $\mathcal{U}$ is an evident additive category with object set $(\mathbb{Z}/3)^2$. Our elaboration of May's results shows that this has a canonical extension over a larger category $\mathcal{V} \supset \mathcal{U}$. More specifically, the category $\mathcal{V}$ has objects u_{ij} (for $i, j \in \mathbb{Z}/3$) and v_k. There are morphisms

$$v_{i+j+1} \xrightarrow{\mu_{ij}} u_{ij} \xrightarrow{\lambda_{ij}} v_{i+j-1},$$

and the full category is generated freely by these, subject only to the relations $\mu_{i+1,j+1}\lambda_{i-1,j-1} = 0$ and $\sum_{i+j=k} \lambda_{ij}\mu_{ij} = 0$. The category $\mathcal{V}$ has only finitely many morphisms, which can be listed explicitly. It also has a rich group of automorphisms. For each morphism in the image of the extended functor $F\colon \mathcal{V} \to \mathcal{D}$, the cofibre can be described as an object in the image of F, or a direct sum of two or three such objects.

Greg Stevenson

We will discuss a framework for proving classification theorems for localizing subcategories of compactly generated triangulated categories using actions of tensor triangulated categories. As both motivation and an application we will discuss the classification of localizing subcategories of the stable derived category of a complete intersection over a field.

An action of a tensor triangulated category T on a triangulated category K is a functor $T \times K \to K$ which is exact and coproduct preserving in each variable and is compatible with the tensor structure on T. In the case that T is rigidly-compactly generated and K is compactly generated this gives rise to a theory of supports which extends the theory of Balmer and Favi and categorifies work of Benson, Iyengar, and Krause.

We will indicate briefly how one constructs the associated support theory and its relationship to the classification problem. This leads to the one of the central abstract results: the local-to-global principle for actions. This theorem allows one to reduce the classification problem to (hopefully) more tractable problems.

As a demonstration of the utility of this machinery we explain recent progress on understanding the lattice of localizing subcategories of the stable derived category of a noetherian separated scheme. In particular, the classification for hypersurfaces in terms of the singular locus will be discussed.

Ivo Dell'Ambrogio

As an illustration of the general methods of relative homological algebra in triangulated categories developed by Christensen, Beligiannis and Meyer-Nest, we explain the following equivariant generalization of two classical, and very useful, results of Rosenberg and Schochet, namely, their universal coefficient and Künneth theorems for complex C*-algebras.

Let G be a finite group. For any G-C*-algebra A, the collection of all H-equivariant topological K-theory groups of A, with H running through all subgroups of G, assemble to form a $\mathbb{Z}/2$-graded Mackey functor $k_*(A)$. The assignment of $k_*(A)$ to A lifts to a stable homological functor defined on the G-equivariant Kasparov category, KK^G. We obtain from this construction, and from the general machinery, the existence of previously unobserved universal coefficient and Künneth spectral sequences abutting to $KK^G(A, B)$ and $K^G(A \otimes B)$, respectively. As input for these spectral sequences we find Ext and Tor groups, respectively, that can be computed in the abelian category of Mackey modules over the complex character ring of G. Convergence – conditional and strong, respectively – is obtained if A is "cellular", in the sense that it belongs to the localizing subcategory of KK^G generated by the algebras $C(G/H)$. Such algebras form a rather large class, containing all commutative G-C*-algebras and closed under the classical "bootstrap" operations.

Jon Carlson

This lecture is a survey of results on endotrivial modules over finite groups and group schemes. Assume that k is a field of characteristic p and G is a finite group or group scheme. A kG-module is endotrivial provided $\mathrm{Hom_k(M, M) \cong M^* \otimes M \cong k \oplus P}$ where P is a projective kG-modules. Thus. M is an endotrivial module if its k-endomorphism rings is a trivial module in the stable category $\mathbf{stmod}(kG)$ of kG-modules modulo projectives. Dade and Puig showed that the endotrivial modules play a substantial role in the block theory of modular group representations. In addition, tensoring with an endotrivial module is a self-equivalence on the stable category. Consequently the group of endotrivial module, which has elements the isomorphism classes of endotrival modules in the stable category and operation given by the tensor product, is an important part of the Picard group of self-equivalences of $\mathbf{stmod}(kG)$. It is group of self-equivalences of Morita type.

Everett Dade introduced the group of endotrivial modules more than thirty years ago. He classified them in the case that G is an abelian p-group. A complete classification of endotrivial modules for all p-groups was achieved a few years ago by Jon Carlson and Jacques Thévenaz building on the work of many others, notably Dade and Jon Alperin. Classifications of the endotrivial modules over other types of groups are included in works by various combinations of Jon Carlson, Dave Hemmer, Nadia Mazza, Dan Nakano, Gabriel Navarro, Geoff Robinson and Thévenaz. Some progress has also been made by Carlson and Nakano on the structure of endotrivial modules for infinitesimal finite group schemes. We review this work in the lecture and also demonstrate a construction of exotic endotrivial modules. The construction relys on the theory of support varieties for kG-modules and connects the work of Paul Balmer and Giordano Favi on gluings in triangulated categories.

Julia Pevtsova

The study of representations of a finite group (scheme) over a field of characteristic p via their restrictions to the subalgebras of the form $k[t]/t^p$, known as cyclic shifted subgroups or π-points, goes back to the work of J. Carlson in the 80's and has an extensive literature base. In this talk, I'll report on the ongoing project joint with J. Carlson and E. Friedlander in which we initiate the study of representations of an elementary abelian p-group E via their restrictions to "rank r shifted subgroups", that is, subalgebras of kE of the form $k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p)$.

In this framework, we introduce various geometric invariants that generalize familiar concepts in the special and well studied case of $r = 1$. In particular, we investigate a somewhat surprising behavior of "r-support varieties" and "nonmaximal r-support varieties" for $r > 1$. Inspired by the concept of a module of constant Jordan type that lead to new constructions of vector bundles on projective spaces, we introduce modules of constant r-socle and r-radical type and show how such a generalization leads to constructing algebraic vector bundles on Grassmannians. The talk will be spiced up with numerous examples of modules with prescribed properties and of corresponding algebraic bundles on Grassmannians.

Srikanth Iyengar

The goal of my talk was to give an introduction to on-going work with Dave Benson and Henning Krause concerning, what we have called, stratification of triangulated categories. The main achievement of our collaboration so far, and the principal driving force behind it, has been a classification of the localizing subcategories of the stable module category of a finite group, in the spirit of Neeman's result for the derived category of a commutative noetherian ring. To this end it has been necessary to develop a theory of local cohomology and derived completion, from commutative algebra, in the context of fairly general triangulated categories. Beyond clarifying what it takes to classify localizing, and also colocalizing, subcategories, these techniques have allowed us to deduce many results that are consequences, but by no means obvious ones, of such classifications. In my talk, I tried to illustrate these by explaining various answers we could obtain for the question: When is there a non-zero morphism between objects in a triangulated category? The talk was based on the following papers, all co-authored with Benson and Krause and available on the math arXiv:

- *Colocalizing subcategories and cosupport*, J. Reine. Angew. Math., to appear.
- *Stratifying triangulated categories*, J. Topology, to appear.
- *Stratifying modular representations of finite groups*, Ann. of Math. **175** (2012); to appear.
- *Local cohomology and support for triangulated categories*, Ann. Sci. École Norm. Sup. (4) **41** (2008), 575–621.

John Greenlees (Joint work with Benson, Hess and Shamir).

The talk described three different characterizations of complete intersections for Noetherian local rings R. The sci condition (regular ring mod regular sequence), the mci condition (eventually multiperiodic resolutions for finitely generate modules), and the gci condition (polynomial growth of Ext algebra). The aim of the talk was to give a homotopy invariant definition, and in particular to give versions of these definitions which apply to $R = C^*(X; k)$ (cochains on a space X).

Before one gets to complete intersections, one needs to deal with the regular case. There are corresponding definitions (s-regular, m-regular and g-regular) for regular local rings. One can make good sense of g-regular (Ext algebra finite dimensional) and then use the putative m-regular condition (finitely generated implies small) to give a *definition* of finite generation. The substitute for the Noetherian condition on a ring R is that there is a map $Q \to R$ from a regular ring, making R a small Q-module. An R-module M is then said to be finitely generated if it is small as a Q-module.

Now, a space is sci if it is obtained as an iterated spherical fibration from a regular space. A space is gci if $H_*(\Omega X)$ has polynomial growth. To make sense of the mci condition, the eventual multiperiodicity states that a Koszul complex is small, but to get equivalence one needs to require that this natural in a suitable sense. So we reach the eci condition which says that iterated cofibrations in the category of bimodules reach a non-trivial small object. The sci condition is stronger than the others, but in the rational and mod p contexts, the gci and eci conditions are equivalent.

The talk also provided some illustrative examples with X the classifying space of a finite group.

Brooke Shipley
For a compact Lie group G of rank r, Greenlees has conjectured that there is an abelian category $A(G)$ of injective dimension r such that the homotopy category of rational G-equivariant cohomology theories is modeled by the derived category of $A(G)$. This conjecture holds for finite groups since rational G-equivariant cohomology theories are just graded rational Mackey functors (which are all injective). This conjecture also holds for $SO(2), O(2)$ and $SO(3)$ by work of Greenlees [13, 14, 15]. In this talk we discussed joint work with Greenlees on this conjecture for G a torus. Specifically we outlined this new, simplified, proof for $SO(2)$ and indicated the changes needed for higher rank tori.

Theorem [19] *For any torus, T^n, there is an abelian category $\mathcal{A}(T^n)$ of sheaves over the space of closed subgroups of T^n and a Quillen equivalence of model categories:*

$$\textit{Rational } T^n\textit{-spectra} \simeq_Q \textit{Rational } dg\mathcal{A}(T^n)$$

Furthermore, $\mathcal{A}(T^n)$ has injective dimension n (the rank.)

In the case of $T = SO(2)$ this Quillen equivalence appears in [29] based on the work in [14]. In [16], Greenlees uses this work to construct $SO(2)$-equivariant elliptic cohomlogy. Our generalization to higher dimensional tori leads to the possibility of constructions of T^g-equivariant cohomology theories associated to complex curves of genus g; see also [17].

One can also specialize to families versions of the above Theorem for rational G-spectra with fixed points concentrated in a given family, $\mathcal{F}$. For example, free T^n-spectra are modeled by differential graded torsion modules over H^*BT^n; this gives a new proof for tori of the results in [18].

Don Stanley Joint work with Adam-Christiaan van Roosmalen
Let R be an abelian category, and $D(R)$ the derived category of R. An aisle $\mathcal{U} \subset D(R)$ is a full subcategory closed under extensions and suspensions, such that the inclusion has a right adjoint. Keller and Vossieck observed that aisles and t-structures are equivalent data. Given an aisle, consider the function

$$\phi(\mathcal{U})\colon \mathbb{Z} \to \{\text{specialization closed subsets of } Spec(R)\}$$

$$n \mapsto \cup_{M \in \mathcal{U}} Supp H_n(M),$$

where $Spec(R)$ is the ideal spectrum, and $Supp$ is the support. It turns out that this function determines the t-structure if we restrict to t-structures in $D^b_{fg}(R)$, the bounded derived category with finitely generated homologies. If R also has dualizing complex, then it is understood which functions can occur, and hence there is a classification of the t-structures.

On the other hand even for slightly more general abelian categories A, such as the coherent sheaves on P^1, the support data of ϕ does not give enough information to determine t-structures. In this talk we introduce other data that replaces the sequences of support that worked in the commutative ring case. This data is a function

$$\phi'(\mathcal{U})\colon \mathbb{Z} \to (W(n), T(n))$$

where $W(n)$ is a wide subcategory of A and $T(n)$ is a tilting torsion theory in the orthogonal of $W(n-1)$ inside $W(n)$. If A is a category of modules over a finite hereditary algebra, then this data classifies t-structures in $D^b_{fg}(A)$.

Fernando Muro The talk is based on joint work with Oriol Raventós (Barcelona).

There are two representability theorems that a compactly generated triangulated category $\mathcal{T}$ may satisfy: Brown's theorem says that any product-preserving functor $\mathcal{T}^{op} \to \mathrm{Ab}$ which takes exact triangles to exact sequences of abelian groups is representable. Adam's theorem says that if $\mathcal{T}^c$ is the category of compact objects in $\mathcal{T}$, any additive functor $(\mathcal{T}^c)^{op} \to \mathrm{Ab}$ which takes exact triangles to exact sequences is the restriction of a representable functor $\mathcal{T}^{op} \to \mathrm{Ab}$, and any natural transformation $\mathcal{T}(-,X)_{|\mathcal{T}^c} \to \mathcal{T}(-,Y)_{|\mathcal{T}^c}$ is represented by a morphism $X \to Y$. These theorems were first proved for the stable homotopy category and later generalized by Neeman, Krause, Franke... to triangulated categories under suitable assumptions.

The classical Adams representability theorem is seldom satisfied. For instance, the derived category of a finite-dimensional hereditary algebra over an uncountable algebraically closed field satisfies Adam's theorem if and only if it is of finite representation type. A failed theorem of Rosický claimed that any well generated triangulated category with a model would asymptotically satisfy a transfinite version of Adams' theorem: for big enough regular cardinals α, if $\mathcal{T}^\alpha$ is the category of α-compact objects in $\mathcal{T}$, any functor $(\mathcal{T}^\alpha)^{op} \to \mathrm{Ab}$ which preserves products of less than α objects and takes exact triangles to exact sequences is the restriction of a representable functor $\mathcal{T}^{op} \to \mathrm{Ab}$, and any natural transformation $\mathcal{T}(-,X)_{|\mathcal{T}^\alpha} \to \mathcal{T}(-,Y)_{|\mathcal{T}^\alpha}$ is represented by a morphism $X \to Y$. Such a result for a given triangulated category would still have amazing consequences, therefore this problem deserves to be studied in specific cases.

In my talk I presented an approach to this problem from the classical point of view of obstruction theory, leading to counterexamples to Rosický's theorem based on recent results by Braun–Göbel and Bazzoni–Šťovíček. These counterexamples are about the representability of natural transformations for $\mathcal{T}$ the derived category of $\mathbb{Z}$ or $\mathbb{C}[x,y]$. I also showed that there would be counterexamples to the representability of functors $(\mathcal{T}^\alpha)^{op} \to \mathrm{Ab}$ if there were a hereditary ring R with α-pure global dimension > 2 for α big enough. The existence of such rings is an open problem.

Amnon Yekutieli

Let A be a noetherian commutative ring, and $\dashv$ an ideal in it. In this lecture I will talk about several properties of the derived $\dashv$-adic completion functor and the derived $\dashv$-torsion functor.

In the first half of the talk I will discuss GM Duality (first proved by Greenlees and May, then treated by Alonso, Jeremias and Lipman), and the closely related MGM Equivalence. The latter is an equivalence between the category of cohomologically $\dashv$-adically complete complexes and the category of cohomologically $\dashv$-torsion complexes. These are triangulated subcategories of the derived category D(Mod A).

I will explain how the derived completion functor can be studied using the concept of $\dashv$-adically projective modules.

In the second half of the talk I will discuss new results: (1) A characterization of the category of cohomologically $\dashv$-adically complete complexes as the right perpendicular to the derived localization of A at $\dashv$. This shows that our definition of cohomologically $\dashv$-adically complete complexes coincides with the original definition of Kashiwara and Schapira. (2) The Cohomologically Complete Nakayama Theorem. (3) A characterization of cohomologically cofinite complexes. This is related to t-dualizing complexes, in the sense of Alonso, Jeremias and Lipman. (4) A theorem on completion by derived double centralizer. This last result extends earlier work of Dwyer-Greenlees and Efimov.

This is joint work with Marco Porta and Liran Shaul.

For full details see the lecture notes
www.math.bgu.ac.il/∼amyekut/lectures/cohom-complete/notes.pdf
or the paper arxiv:1010.4386 .

Anthony Licata

A *categorical sl_2 action* consists of a sequence of k-linear additive categories $\mathrm{D}(-N), \dots, \mathrm{D}(N)$ together with functors, for $\ell \in \mathbb{Z}$,

$$\mathrm{E}(\ell) : \mathrm{D}(\ell-1) \to \mathrm{D}(\ell+1) \text{ and } \mathrm{F}(\ell) : \mathrm{D}(\ell+1) \to \mathrm{D}(\ell-1)$$

satisfying the relations

$$\mathrm{E}(\ell-1) \circ \mathrm{F}(\ell-1) \cong \mathrm{id}_{\mathrm{D}(\ell)}^{\oplus \ell} \oplus \mathrm{F}(\ell+1) \circ \mathrm{E}(\ell+1) \text{ if } \ell \geq 0. \tag{1}$$

$$\mathrm{F}(\ell+1) \circ \mathrm{E}(\ell+1) \cong \mathrm{id}_{\mathrm{D}(\ell)}^{\oplus -\ell} \oplus \mathrm{E}(\ell-1) \circ \mathrm{F}(\ell-1) \text{ if } \ell \leq 0 \tag{2}$$

On the complexified (split) Grothendieck groups $V(\ell) := K(\mathrm{D}(\ell)) \otimes_{\mathbb{Z}} \mathbb{C}$, the functors $\mathrm{E}(\ell)$ and $\mathrm{F}(\ell)$ induce maps of vector spaces $e(\ell) := [E(\ell)]$ and $f(\ell) := [F(\ell)]$. By the above relations, $V = \oplus V(\ell)$ is a locally finite representation of $sl_2(\mathbb{C})$, where $e = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ acts by $\oplus_\ell e(\ell)$ and $f = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ acts by $\oplus_\ell f(\ell)$.

Any such sl_2 action on $\oplus_\ell V(\ell)$ integrates to an SL_2 action. Hence the reflection element

$$t = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

induces an isomorphism of vector spaces $V(\ell) \xrightarrow{\sim} V(-\ell)$. An interesting question is whether this isomorphism can be lifted to an equivalence of categories $\mathrm{T} : \mathrm{D}(\ell) \to \mathrm{D}(-\ell)$ coming from a lift of the reflection element.

In their seminal work, Chuang and Rouquier constructed such an equivalence under the assumption that the underlying weight space categories are abelian, and the categorical sl_2 action is by exact functors. More precisely, they constructed an equivalence between the corresponding derived categories.

In our motivating geometric examples, however, the natural functors are not exact, and as a result the weight space categories $\mathrm{D}(\ell)$ are triangulated instead of abelian. Nevertheless, it is still natural to ask for an equivalence between $\mathrm{D}(\ell)$ and $\mathrm{D}(-\ell)$.

The setup we use in order to modify the Chuang-Rouquier construction is that of *strong categorical sl_2 actions*. In particular, this means that categories $\mathrm{D}(\ell)$ are graded and the Es and Fs satisfy a graded version of relations (1) and (2). The grading implies that on the level of the Grothendieck group there is an action of the quantum group $U_q(sl_2)$.

As an additional data, we demand functors

$$\mathrm{E}^{(r)}(\ell) : \mathrm{D}(\ell - r) \to \mathrm{E}^{(r)}(\ell + r) \text{ and } \mathrm{F}^{(r)}(\ell) : \mathrm{D}(\ell + r) \to \mathrm{D}(\ell - r)$$

which categorify the elements $e^{(r)} = \frac{e^r}{r!}$ and $f^{(r)} = \frac{f^r}{r!}$ along with natural transformations X, T which are used to rigidify the isomorphisms (1) and (2).

Here is a sketch of the construction. If we restrict $t \in SL_2$ to the weight space $V(\ell)$ with $\ell \geq 0$, then one can write t as

$$t = f^{(\ell)} - f^{(\ell+1)} e + f^{(\ell+2)} e^{(2)} \pm \dots$$

where the sum is finite since $V(\ell) = 0$ for $\ell \gg 0$. We can lift $f^{(\ell+s)} e^{(s)}$ to the composition of functors $\mathrm{F}^{(\ell+s)} \circ \mathrm{E}^{(s)}$. We then form a complex of functors Θ_* whose terms are

$$\Theta_s := \mathrm{F}^{(\ell+s)}(s) \circ \mathrm{E}^{(s)}(\ell + s)\langle -s \rangle.$$

(the $\langle \cdot \rangle$ represents a shift in the grading). The connecting maps of the complex are defined via various adjunction morphisms.

The main result of the talk states that if our categories $\mathrm{D}(\ell)$ are triangulated then the convolution of this complex is an equivalence $\mathrm{T} : \mathrm{D}(\ell) \xrightarrow{\sim} \mathrm{D}(-\ell)$.

Sabin Cautis

In 2004 Chuang and Rouquier proved a version of Broué's abelian defect group conjecture for the symmetric group by introducing the notion of a strong categorical $sl(2)$ action. Their key idea was to show that the Weyl involution of $sl(2)$ can be lifted to give an equivalence of categories.

More recently, in joint work with Joel Kamnitzer and Anthony Licata, we constructed similar categorical $sl(2)$ actions on (derived) categories of coherent sheaves on cotangent bundles to Grassmannians $DCoh(T^*G(k, N))$. Furthermore we showed that such actions also induce equivalences and thus were able to construct a natural equivalence $DCoh(T^*G(k, N)) \simeq DCoh(T^*G(N-k, N))$ generalizing earlier work of Namikawa and Kawamata on derived equivalences in algebraic geometry.

More recently, in joint work with Joel Kamnitzer, we extend this notion to define categorical $sl(n)$ actions. We show that such an action induces an action of the braid group on n strands (this should be thought of as a cover of the symmetric group which is the Weyl group of $sl(n)$). By constructing such an $sl(n)$ action on (derived) categories of coherent sheaves on n-step partial flag varieties we obtain a braid group action on them.

In this talk I will explain what is a categorical $sl(n)$ action and how it can be used to construct braid group actions. This definition will be motivated by braid group actions constructed via Seidel-Thomas spherical twists. I will illustrate all of this by defining an $sl(3)$ action on the resolution of $\mathbb{C}^2/\mathbb{Z}_3$, thus obtaining the well known braid group action on (the derived category of) this space.

This method of constructing braid group actions has applications to geometry (as described above), knot theory (e.g. algebro-geometric constructions of Khovanov homology) and representation theory of finite group (e.g. the work of Chuang and Rouquier). It is also natural to ask if there are similar constructions for other groups, for instance, mapping class groups.

Wendy Lowen
Compact generation of triangulated categories was introduced by Neeman. One of the motivating situations is given by "nice" schemes (i.e. quasi-compact separated schemes, later extended to quasi-compact quasi-separated schemes by Bondal and Van den Bergh). The ideas of the proofs later crystallized in Rouquier's (co)covering theorem which describes a certain covering-by-Bousfield-localizations situation in which compact generation (later extended to α-compact generation by Murfet) of a number of "smaller pieces" entails compact generation of the whole triangulated category. The notions needed in the (co)covering concept can be interpreted as categorical versions of standard scheme constructions like unions and intersections of open subsets, and in the setup of Grothendieck categories rather than triangulated categories they have been important in non-commutative algebraic geometry. In this talk we present a (co)covering theorem for Grothendieck categories based upon these notions, which can be used to prove compact generation of the derived category of certain Grothendieck categories. As such, the result follows from the triangulated (co)covering theorem, and it implies Neeman's original result for schemes by application to the Grothendieck category of quasi-coherent sheaves. Our interest in such an intermediate result comes from its applicability to Grothendieck categories that originate as "non-commutative deformations" of schemes, more precisely abelian deformations of categories of quasi-coherent sheaves in the sense of Lowen and Van den Bergh. In general, there remain obstructions to the lifting of compact generators of some of the relevant categories under deformation. On the other hand, in some situations (eg. for quasi-projective surfaces) we can use the (co)cover theorem in combination with techniques of Keller and Lowen to obtain compact generation of all deformations.

Daniel Murfet
Functors between triangulated categories are often described in terms of integration against a kernel. In the categorical setting this integration takes the form of a pushforward. I will describe a new approach to constructing finite models of pushforwards for matrix factorisations and some applications including, if time permits, computations of homological link invariants.

References

[1] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[2] Paul Balmer. Supports and filtrations in algebraic geometry and modular representation theory. *Amer. J. Math.*, 129(5):1227–1250, 2007.

[3] Paul Balmer. Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings. *Algebr. Geom. Topol.*, 10(3):1521–1563, 2010.

[4] Paul Balmer and Giordano Favi. Gluing techniques in triangular geometry. *Q. J. Math.*, 58(4):415–441, 2007.

[5] Silvana Bazzoni and Jan Stovicek. On the abelianization of derived categories and a negative solution to Rosicky's problem.

[6] David J. Benson, Srikanth Iyengar, and Henning Krause. Local cohomology and support for triangulated categories. *Ann. Sci. École Norm. Sup. (4)*, 41:573–619, 2008.

[7] David J. Benson, Srikanth Iyengar, and Henning Krause. Stratifying modular representations of finite groups. To appear in Annals of Math.

[8] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007.

[9] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008.

[10] Tom Bridgeland. Stability conditions and Kleinian singularities. *Int. Math. Res. Not. IMRN*, (21):4142–4157, 2009.

[11] Carles Casacuberta and Amnon Neeman. Brown representability does not come for free. *Math. Res. Lett.*, 16(1):1–5, 2009.

[12] Duiliu-Emanuel Diaconescu. Enhanced D-brane categories from string field theory. *J. High Energy Phys.*, (6):Paper 16, 19, 2001.

[13] J. P. C. Greenlees. Rational $O(2)$-equivariant cohomology theories. *Fields Inst. Comm.* 19 (1998), 103-110.

[14] J. P. C. Greenlees. Rational S^1 Equivariant Stable Homotopy Theory. *Mem. Amer. Math. Soc.* 138 (1999), no. 661.

[15] J. P. C. Greenlees. Rational $\mathrm{SO}(3)$-equivariant cohomology theories. *Homotopy methods in algebraic topology* (Boulder, CO, 1999), 99–125, Contemp. Math., 271, Amer. Math. Soc., Providence, RI, 2001.

[16] J. P. C. Greenlees. S^1-equivariant elliptic cohomology. *Topology* 44 (2005), no. 6, 1213–1279.

[17] J. P. C. Greenlees Algebraic groups and equivariant cohomology theories. *Elliptic cohomology,* 89–110, London Math. Soc. Lecture Note Ser., 342, Cambridge Univ. Press, Cambridge, 2007.

[18] J. P. C. Greenlees and B. Shipley. An algebraic model for free rational G-spectra for connected compact Lie groups G. To appear in Math. Z.

[19] J. P. C. Greenlees and B. Shipley Algebraic models for rational torus-equivariant cohomology theories. arXiv:1101.2511, January 2011.

[20] Kentaro Hori and Johannes Walcher. D-branes from matrix factorizations. *C. R. Phys.*, 5(9-10):1061–1070, 2004. Strings 04. Part I.

[21] Anton N. Kapustin and Yi Li. D-branes in Landau-Ginzburg models and algebraic geometry. *J. High Energy Phys.*, (12):005, 44 pp. (electronic), 2003.

[22] Anton N. Kapustin and Dmitri O. Orlov. Lectures on mirror symmetry, derived categories, and D-branes. *Uspekhi Mat. Nauk*, 59(5(359)):101–134, 2004.

[23] Valery A. Lunts and Dmitri O. Orlov. Uniqueness of enhancement for triangulated categories. *J. Amer. Math. Soc.*, 23(3):853–908, 2010.

[24] Ralf Meyer and Ryszard Nest. The Baum-Connes conjecture via localisation of categories. *Topology*, 45(2):209–259, 2006.

[25] Fernando Muro, Stefan Schwede, and Neil Strickland. Triangulated categories without models. *Invent. Math.*, 170(2):231–241, 2007.

[26] Amnon Neeman. Brown representability follows from Rosický's theorem. *J. Topol.*, 2(2):262–276, 2009.

[27] Jiří Rosický. Generalized Brown representability in homotopy categories. *Theory Appl. Categ.*, 14:no. 19, 451–479 (electronic), 2005.

[28] Stefan Schwede. The stable homotopy category is rigid. *Ann. of Math. (2)*, 166(3):837–863, 2007.

[29] B. Shipley. An algebraic model for rational S^1-equivariant homotopy theory. *Quart. J. Math* 53 (2002), 87–110.