Report on Research in Teams Project: Universal Higher Extensions

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We benefited greatly from the *Research in Teams* workshop at the Banff International Research Station. To explain this, we outline below the before/after-effect which resulted from the opportunity to meet face-to-face and to devote ourselves entirely to the task at hand for the duration of the visit.

1 The Situation Before the Visit

In one stream of development, one partner in the project had just achieved an interpretation of group cohomology or Lie-algebra cohomology in terms of higher-dimensional central extensions, as developed by Rodelo–Van der Linden [14, 15]. It extends the classical interpretation of the second group cohomology in terms of equivalence classes of short exact sequences with central kernel. This development builds upon the notion of *semi-abelian category* as in Janelidze–Márki–Tholen [11], and the concept of *higher central extension* developed within the framework of categorical Galois theory based on the work of Janelidze et al. (See [2, 5, 9, 10].) In addition, methods from the theory of simplicial groups are used.

In a parallel and complementary stream of development, the second partner in the project had just achieved a proof of existence of universal n-step extensions of modules over an arbitrary unitary ring [12]. This development involves certain higher torsion theories, some potentially noncommutative. It immediately has several applications ranging from

- a torsion theoretic conceptual hindsight explanation for existing computational results about the effect of plus-constructions on the homotopy groups of a space, to
- identifying those groups, respectively Lie-algebras, which have a universal central n-extension in the sense of Rodelo–Van der Linden, to
- speculation about probable analogues in other non-abelian categories, notably the category of Π-algebras.

While already a superficial interface between these two developments gives rise to exciting insights, a lot of questions remained unanswered (e.g. how about a constructive description of these abstractly existing universal central n-extensions?). Moreover, as hinted above, a number of further developments appeared promising, but there seemed to be no natural occasion for collaboration. It is this situation which lead us to apply for a *Research in Teams* workshop at BIRS.

2 Scientific Progress Made

The workshop provided the opportunity for a blend of mutual education and extensive brain storming. As a consequence we now have clear outlines of several projects which are motivated by the two separate developments outlined above. Here is a brief description of these projects.

2.1 Comonad derived coefficient functors in semi-abelian varieties

A coefficient functor on a semi-abelian variety $\mathcal{B}$ is detemined by a Birkhoff subcategory $\mathcal{A}$ and its reflector $I\colon \mathcal{A} \to \mathcal{B}$. We describe these derived functors uniformly using a composite of 'abelianization', followed by 'tensoring over a solid ring'. We hope to describe the composite using a suitable generalization of the non-abelian tensor product introduced by Brown and Loday [3]; the work of Hartl [6] will be helpful here.

2.2 A Quillen model structure for categorical Galois theory?

Here we use categorical Galois theory to define higher order central extensions. We discovered a mechanism by which this kind of categorical Galois theory interfaces with an associated Quillen model category structure [13]. This interface itself is quite unexpected and exciting. At a technical level it provides a nice conceptual context for recent work on the long exact homology sequence of a extension in a purely semi-abelian setting; see Everaert [4]. – Work on the relationship between this long exact sequence and the Serre spectral sequence of the extension is in progress.

2.3 Homological looping in semi-abelian categories

... is possible via projective presentations. The only problem is that such a projective presentation itself does not lie in the same category as the original object. When the context is suitably enlarged though, the concept makes sense and acts as it should.

2.4 The derived Yoneda lemma

In module categories Yoneda's lemma has derived versions; see [16, 7, 8]. This result is essential in characterizing higher torsion modules associated to a right exact reflector and, hence, in identifying those modules which are covered by a higher order universal extension; see [12].

However, it is clear that the validity of derived versions of Yoneda's lemma is not restricted to module categories at all. We are expecting such higher versions in more general settings as well, including settings of semi-abelian varieties, where the resulting Ext-groups should classify *higher central extensions* as discussed in [15].

3 Outcome of the Meeting

We are extremely grateful for the opportunity to work in the *Research in Teams*-setting which BIRS offers to the mathematical community. We found the working environment at the Banff Centre to be ideal; it allows a period of complete devotion to scientific investigation and collaboration—which is precisely what we needed to get our joint project started.

As explained above, the impact of the week at BIRS on our work is huge. It enabled us to reach far beyond our initial scope, and to develop more of the combined potential of previous works and quite dissimilar backgrounds. We were able to build the foundation for a fruitful collaboration, and this collaboration already gave rise to important new insights in our field.

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