Heights on moduli space for post-critically finite dynamical systems

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1 Overview of the Field

The purpose of this Research In Teams event was to consider the arithmetic properties of post-critically finite (PCF) rational maps. In the study of complex holomorphic dynamics, it is a general theme that the dynamical properties of a holomorphic map are largely determined by the behaviour of the critical points. In studying the dynamics of a rational map, then, one is lead to consider the orbits of the critical points, and maps for which these critical orbits are all finite gain special prominence. These are the PCF maps. Let $\mathcal{M}_d$ denote the moduli space of degree-d endomorphisms of $\mathbb{P}^1$, up to change of variables. If $d = m^2$, certain PCF elements of $\mathcal{M}_d$ stand out, namely the so-called Lattès examples. These are maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that there is an elliptic curve E and an integer m, such that f is the action induced from $[m]$ by viewing $\mathbb{P}^1$ as the Kummer surface of E. If $\mathcal{L}_d \subseteq \mathcal{M}_d$ is the locus of these Lattès examples, then one expects PCF maps to be somewhat sparse in $\mathcal{M}_d \setminus \mathcal{L}_d$. A deep result of Thurston makes this more concrete.

Theorem 1 (Thurston [3], see also [4]) *The PCF points in $\mathcal{M}_d \setminus \mathcal{L}_d$ is contained in a countable union of 0-dimensional subvarieties.*

Note that, since $\mathcal{M}_d \setminus \mathcal{L}_d$ and the subvarieties in Thurston's result are all defined over $\mathbb{Q}$, all non-Lattès PCF maps have (up to change of coordinates) algebraic coefficients. Thurston's result, on the other hand, does not preclude the possibility of most (or even all) rational functions over $\overline{\mathbb{Q}}$ being PCF, however unlikely this eventuality might seem.

From an arithmetic perspective, a refinement of the theorem above would be given by the following.

Conjecture 1 *For some height function $h_{\mathcal{M}_d}$ relative to an ample class, the set of PCF points in $\mathcal{M}_d \setminus \mathcal{L}_d$ is a set of bounded height.*

Note that Theorem 1 would follow from Conjecture 1. Although Thurston's results in [3] contain more information than the statement of Theorem 1, the methods of proof involve a deep examination of iteration on Teichmuller space, and it would be of interest to have an algebraic proof of Theorem 1 Indeed, Conjecture 1 implies something much stronger than Theorem 1: that given any constant C, the set of PCF rational maps of degree d and coefficients of algebraic degree at most C is, up to change of variables, finite and effectively computable. This is a far cry from the possiblity of all algebraic maps being PCF, which is not precluded by Theorem 1, or the more subtle results of Thurston given in [3].

Silverman has proposed a refined version of Conjecture 1. We define the *critical height* on $\mathcal{M}_d(\overline{\mathbb{Q}})$ by

$$h_{\mathrm{crit}}(f) = \sum_{\beta\in\mathbb{P}^1}(e_f(\beta)-1)\hat{h}_f(\beta),$$

where $e_f(\beta)$ is the index of ramification of f at β, and $\hat{h}_f(\beta)$ is the canonical height (as defined in [1]). As $\hat{h}_f(\beta)$ vanishes precisely if β has a finite forward orbit under f, it follows that h_{crit} vanishes precisely on PCF points. What is not clear is that the critical height is in any way related to any height function on $\mathcal{M}_d$, in the usual sense.

Conjecture 2 *For some height function $h_{\mathcal{M}_d}$ relative to an ample class, there exist constants $c_1, ...c_4$ such that*

$$c_1 h_{\mathcal{M}_d}(f) - c_2 \leq h_{\mathrm{crit}}(f) \leq c_3 h_{\mathcal{M}_d}(f) + c_4.$$

Conjecture 1 is equivalent to the assertion that $h_{\mathcal{M}_d}$ is bounded on the subset defined by $h_{\mathrm{crit}} = 0$, and so is clearly weaker than Conjecture 2. Although both of these conjectures are quite strong, there is some hope of progress. In particular, one of the participants proved Conjecture 2 for the moduli space of polynomials [2] in 2010. Simultaneously, Epstein [5] used related arithmetic techniques to prove various results on PCF polynomials which, while not as strong as the results in [2] from the perspective of Conjecture 2, allowed a proof of some of the more subtle parts of Thurston's work omitted in the statement of Theorem 1 above.

2 Scientific Progress Made

The main goal of the Research in Teams workshop was to extend some of the earlier work, most notably the results in [2], to the context of rational functions. This seems somewhat difficult, as the theory of local heights (used extensively in [2]) is greatly simplified in the case of polynomial dynamics. Employing geometric results of McMullen [6], we were able to reduce Conjecture 1 to the following plausible conjecture in non-archimedean dynamics

Conjecture 3 *Every sufficiently attracting fixed point of a rational map is the accumulation point of a critical orbit. Specifically, if $|\cdot|_v$ is a norm on $\overline{\mathbb{Q}}$ extending the p-adic norm on $\mathbb{Q}$, then there is a constant $C_{p,d} \leq 1$ such that if $f(z) \in \overline{\mathbb{Q}}(z)$ has degree d, if $f(\gamma) = \gamma$, and if $|f'(\gamma)|_v < C_{p,d}$, then γ is the accumulation point of critical orbit. We further require, for each fixed $d \geq 2$, that $C_{p,d} = 1$ for all but finitely many primes p.*

The analogue of Conjecture 3 for complex dynamics has been known since the time of Fatou, and so it seems reasonable to posit the same claim for p-adic dynamics. Moreover, we were able to establish this conjecture, and hence Conjecture 1, in several restricted contexts.

Theorem 2 *Conjecture 3 holds for rational maps of degree 2, and rational maps of degree 3 with a ramified fixed point. In particular, Conjecture 1 holds for $\mathcal{M}_2$.*

While it might appear that the above theorem also proves Conjecture 1 for certain maps of degree 3, this is not quite true; one requires a result for maps of degree 9 to obtains this. It should be noted that the restriction to a certain subvariety of $\mathcal{M}_3$ was largely a practical concern, and in discussions with Alon Levy since the conclusion of the workshop, we seem to have eliminated this restriction. It is not clear that the argument generalizes, but we remain hopeful that Conjecture 1 will be proven eventually (and recent work with Levy make this seem increasingly likely). It is also interesting to note that we were able to establish the conjecture in another context.

Theorem 3 *Conjecture 3 holds for the moduli space of polynomials of degree d, for any $d \geq 2$.*

Although this result is of lesser interest, since Conjecture 1 (and, indeed, the much stronger Conjecture 2) is already known in this context, the proof in [2] is of a very different nature, unrelated to Conjecture 3. Theorem 3, then, provides further evidence for the veracity of Conjecture 3, as well as allowing for several new results in the p-adic dynamics of polynomials, analogous to some classical results in complex dynamics. Although the main conjecture addressed at the workshop remains open, we expect that these results will lead to a significant publication, and the progress made has brought us closer to our ultimate goals of proving Conjectures 1 and 2.

References

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