Multivariable Complex Dynamics

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1 Introduction

Broadly speaking, the object in 'multivariable complex dynamics' is to understand dynamical properties of holomorphic (and more generally meromorphic) mappings $f : X \to X$ on compact complex manifolds X with complex dimension two or larger. The subject has been actively pursued for about twenty years now, and it has its genesis in several sources.

First among these is the remarkable success in understanding dynamics of (single variable) holomorphic maps $f : \mathbf{P}^1 \to \mathbf{P}^1$ on the Riemann sphere, and in particular of polynomial maps $f : \mathbf{C}^2 \to \mathbf{C}^2$. In the 1980's it was noted [42] that the Hénon mappings $(x, y) \mapsto (y, y^2 + c - ax)$ are natural two dimensional generalizations of quadratic polynomials $z \mapsto z^2 + c$ of a single variable. More generally, people began to ask whether and how certain natural and well-understood concepts for holomorphic maps of one variable might generalize to higher dimensions. For example, people wondered what the appropriate definition of Fatou and Julia (i.e. stable and unstable) set might be for a holomorphic map $f : \mathbf{P}^n \to \mathbf{P}^n$ on higher dimensional projective space [37]; similarly, there were questions about the local behavior of a multi-variable map near a superattracting or indifferent fixed point.

Somewhat apart from such considerations, there was a remarkable and simple argument of Gromov [41] that gave the precise value for the topological entropy of a holomorphic map $f : \mathbf{P}^n \to \mathbf{P}^n$ in terms of the degrees of the homogeneous polynomials defining f. The variational principle from smooth dynamics states that topological entropy is the supremum of the various metric entropies with respect to f-invariant measures. Hence it was logical to investigate questions concerning existence, uniqueness and structure of f-invariant measures that maximize the metric entropy of a holomorphic map f.

Finally, it was realized that recent developments in several complex variables and dynamical systems, particularly those surrounding pluripotential theory and the theory of non-uniformly hyperbolic systems, might furnish sufficient technical tools for addressing dynamical issues like the ones described above.

Two decades away from its beginning, the field of multivariable complex dynamics continues to flourish and grow steadily. Some of the initial questions (e.g. existence and uniqueness of measures of maximal entropy) are much better understood now, having been answered for broad classes of holomorphic mappings. Others have seen less definitive progress, and of course, the list of questions that motivated the subject initially has greatly expanded. So too has the range of techniques that are employed to address them. In particular, complex geometry now plays a large role. Some of the more recently employed tools (e.g. laminar currents, superpotentials, and valuative analysis) did not exist or were not nearly as refined when the subject began. Rather, they were developed in direct response to dynamical questions, and now seem likely to find application in the larger world of mathematics outside complex dynamics.

Our goal in the remainder of this exposition is to describe some of these developments, particularly those related to our March 2009 conference at Banff International Research Station, in more detail below. Though we do not for the most part attempt to present things in a talk-by-talk order, the great majority of what follows was exposed and elaborated upon by one or more of the speakers at the conference. We note in passing that two of our scheduled speakers, Igor Dolgachev and Francois Berteloot, were unable to attend the meeting because of last minute emergencies. Fortunately we were able to prevail upon a couple of other conference participants, Sarah Koch and Chris Lipa, to prepare and give talks on short notice. It is no overstatement on our part to say that the facilities and staff of BIRS were wonderful throughout our meeting and that the success of the meeting was in no small part their accomplishment as well as that of the attending mathematicians.

2 Measures of maximal entropy

Gromov [41] gave a very elegant argument that the topological entropy of a holomorphic mapping $f : \mathbf{P}^n \to \mathbf{P}^n$ is equal to $n \log d$ where d is the degree of the homogeneous polynomials defining f. His reasoning easily extends to give a formula for the entropy of a holomorphic mapping on any compact Kähler manifold. With somewhat more effort, Dinh and Sibony [29] have pushed further, obtaining

Theorem 2.1 *Let $f : X \to X$ be a meromorphic mapping on a compact Kähler manifold X, ω with $\dim_{\mathbf{C}} X = N$. Then the topological entropy of f is no larger than $\max_{1 \le j \le N} \log \lambda_j$, where the j*th dy*namical degree λ_j of f may be defined by*

$$\lambda_j = \lim_{n\to\infty} \left(\int f^{n*}\omega \wedge \omega^{n-j} \right)^{1/n}.$$

We recall that a meromorphic map $f = \pi_2 \circ \pi_1^{-1}$ is given by its 'graph' $\Gamma \subset X \times X$, an irreducible subvariety whose projection $\pi_1 : \Gamma \to X$ onto the first factor is a birational morphism. If π_1 collapses some hypersurface in Γ onto a lower dimensional subvariety $I(f) \subset X$, it follows that f will be ill-defined at every point of $I(f)$, sending such points onto positive dimensional subvarieties of X. Consequently many things things that one takes more or less for granted for smooth maps f (e.g. the definitions of topological entropy and pullback of a form by f) require additional interpretation in the meromorphic setting. Mostly, we will ignore such issues in this brief survey, focusing only on unresolved points of interest.

When the mapping f in Theorem 2.1 is holomorphic, a deep theorem of Yomdin [48] implies that the upper bound in the theorem is actually an equality. In the meromorphic setting, however, it is not known whether Yomdin's theorem applies. Indeed Guedj [39] has given a simple example of a meromorphic map on a compact Kähler surface whose entropy is zero despite the fact that the right side of the inequality in Theorem 2.1 is positive. It remains an interesting open question then to find necessary and sufficient conditions on a meromorphic map guaranteeing that the upper bound in the theorem is actually an equality.

Almost all work toward answering this question has taken the same basic approach, which relies on two ideas. First of all, the topological entropy of a meromorphic mapping dominates the metric entropy of the mapping relative to any particular invariant measure, so to establish equality in Theorem 2.1, it suffices to produce an f-invariant measure μ with metric entropy $h_\mu(f) = \max \log \lambda_j$. Secondly, in the algebraic setting $X \hookrightarrow \mathbf{P}^n$, the powers ω^j of the Kähler form are in some sense averages over intersections $X \cap L$ with codimension j linear subspaces. Hence one might hope to somehow construct a natural f-invariant measure by pulling back subvarieties $V \subset X$ with codimension J chosen so that λ_J is maximal.

In the particular case $J = N := \dim X$, λ_N is just the topological degree of and one seeks to construct a measure μ of maximal entropy by pulling back points. This has in fact been accomplished by Guedj [40] relying on important ideas of Briend and Duval [14] who considered the particular case of holomorphic maps on $\mathbf{P}^n$.

Theorem 2.2 *Suppose that $f : X \to X$ is a holomorphic map on a compact Kähler manifold such that λ_N uniquely maximizes the dynamical degrees of f. Then there exists an f-invariant and mixing probability measure μ on X given by*

$$\mu = \lim_{n\to\infty} \frac{1}{\lambda_N^n} \sum_{f^n(q)=p} \delta_q$$

for any $p \in X$ outside some pluripolar exceptional set. The entropy of μ is $\log \lambda_N$.

In complex dimension two, λ_1 and λ_2 are the only relevant dynamical degrees, and Theorem 2.2 therefore speaks to the case $\lambda_2 > \lambda_1$ of *large topological degree*. The case $\lambda_1 < \lambda_2$ of *small topological degree* is more complicated and has been studied [23–25]. Among other things, the results in those papers imply

Theorem 2.3 *Let $X \hookrightarrow \mathbf{P}^\ell$ be a complex projective surface and $f : X \to X$ a meromorphic map with small topological degree. Suppose additionally that*

- $(f^n)^*\omega = (f^*)^n\omega$ *for all $n \in \mathbf{N}$ and ω the Kähler form on X;*
- *f has finite dynamical energy.*

Then there is an f-invariant and mixing probability measure μ on X given by

$$\mu = \lim_{m,n\to\infty} \frac{1}{\lambda_1^{m+n}} f^{n*}H_1 \wedge f_*^m H_2$$

for generic hyperplane sections $H_1, H_2 \subset X$. The entropy of μ is $\log \lambda_1$.

The wedge product in the displayed equation is interpreted as the measure obtained by equidistributing point masses over the intersection points (counted with multiplicity) of the two curves. It is not known whether either of the two displayed hypotheses can be satisfied by all meromorphic surface maps with small topological degree, though they seem to be satisfied by reasonably generic maps. We will return to the first hypothesis later, focusing instead on the 'finite dynamical energy' condition now. The first step in proving Theorem 2.3 is to show that there are positive closed $(1,1)$ currents T^+ and T^- such that

$$T^+ = \lim_{n\to\infty} \frac{f^{n*}H}{\lambda_1^n}, \quad T^- = \lim_{n\to\infty} \frac{f_*^n H}{\lambda_1^n}$$

for generic hyperplane sections H. Proving this requires only the first displayed hypothesis. Finite dynamical energy is a condition that allows one to reasonably interpret the product $T^+ \wedge T^-$ as the desired probability measure μ. It is shown in [24] that if the condition is violated then the product $T^+ \wedge T^-$ tends to be quite degenerate. Nevertheless, it remains an interesting open problem to show that $T^+ \wedge T^-$ is a well-defined probability measure in all circumstances, either by showing that all small degree maps have finite dynamical energy or finding a definition of the wedge product that is better suited to the dynamical context.

In the context of Theorems 2.2 and 2.3, one can say a good deal more about the properties of the measure μ of maximal entropy. For instance, in Theorem 2.2 μ is a non-uniformly hyperbolic measure of repelling type, and plurisubharmonic functions are locally integrable with respect to μ (which implies among other things that the $\log \|Df\|$ is integrable and therefore Lyapunov exponents are well-defined). Moreover, μ records the asymptotic distribution of repelling periodic points for f. In Theorem 2.3, it is not clear whether $\log \|Df\|$ is generally integrable with respect to the measure μ. However, assuming this integrability, one has that μ is non-uniformly hyperbolic of saddle type, and that it records the distribution of saddle periodic points of f.

Even without the integrability condition, the measure μ in Theorem 2.3 has a certain geometric product structure that it inherits from the currents T^+ and T^-. The current T^+ turns out to be what is known as *laminar*. That is, very roughly speaking, it has an almost everywhere local structure given by averaging a family of disjoint disks with respect to a transverse measure. The notion of laminarity was introduced in [13] and subsequently refined by Dujardin (e.g. [31, 32]), DeThelin (e.g. [18, 19]) and others. The current T^- is not laminar but rather *woven*, which is a weaker analogue of laminarity in which disks are allowed to cross each other. It is shown in [25] that after lifting the measure μ to the natural extension of the (possibly non-invertible) dynamical system $f : X \to X$, one can exhaust μ from below by a sum of product measures obtained by intersecting the families of disks that define T^+ with the families that define T^-.

For meromorphic maps in dimension larger than two, less is known about existence and properties of the measure of maximal entropy outside the 'large topological degree' case. Nevertheless, there have been many interesting recent developments. Motivated by dynamical considerations in higher dimensions, Dinh and Sibony [30] have developed a theory of 'superpotentials' that is well-suited for dealing with positive closed currents with bidegree (i.e. codimension) larger than one. Relying on this theory, DeThelin and Vigny [21] have constructed invariant measures of maximal entropy for generic birational maps of $\mathbf{P}^k$. A key ingredient in this work is an adaptation of a deep result of Yomdin. Roughly speaking, it says that if one considers the appropriate analogue of the sequence of probability measures used to define μ in Theorem 2.3, and if one can extract a limit point μ of this sequence that is regular enough to permit application of Pesin's theory of non-uniform hyperbolicity, then the entropy of μ is maximal. Concerning further properties of such measures, another recent result of DeThelin [19] implies that if μ also ergodic, and f has a unique maximal dynamical degree, then the Lyapunov exponents are bounded away from zero solely in terms of the dynamical degrees of f.

An interesting feature of the work of DeThelin and Vigny is that it does not require a detailed 'product structure' type understanding of the invariant measures they consider. This is important, because for maps on surfaces, inferring product structure depended on the notion of laminarity for positive closed $(1, 1)$ currents. At present, there does not seem to be a higher dimensional theory of laminarity that is sufficiently developed for dynamical purposes, though we mention that the paper [?] by Dinh represents some progress in this direction.

In closing this section, we remark that the measure of maximal entropy is only one of many potentially interesting measures to consider for meromorphic maps. For instance, the measure of maximal entropy for a holomorphic endomorphism $f : \mathbf{P}^2 \to \mathbf{P}^2$ records the distribution of repelling periodic points, but it is also interesting to understand measures that might arise as limits of saddle cycles. Diller and Jonsson and, more recently, DeThelin [?] have done some work in this direction, but little is known in general.

3 Geometry and classification of meromorphic self-maps

A holomorphic map $f : \mathbf{P}^1 \to \mathbf{P}^1$ is roughly classified by a single number, its topological degree. Moreover, this number is equal to the degree of the homogeneous polynomials that define f, and it is therefore easily determined. The situation in higher dimensions is far more subtle and interesting. As noted earlier, if $f : X \to X$ is a meromorphic map on a compact Kähler manifold X of dimension n, then there are n relevant 'dynamical degrees' $\lambda_1, \ldots, \lambda_n$ to consider. These cannot generally be read off easily from formulas defining f.

Perhaps the first difficulty one encounters in this connection is that of *algebraic stability*, which was recognized first by Fornæss and Sibony [37]. Namely, the dynamical degrees are defined by pulling back a power of the Kähler form by iterates of f, but the pullback operator need not behave well under iteration. That is, it is not necessarily true that $(f^n)^*\omega^k = (f^*)^n\omega^k$ for all $n \in \mathbf{N}$. Fornæss and Sibony called a meromorphic map *algebraically stable* if it satisfied this condition for $k = 1$. Let us here call f p-stable if it satisfies the same condition when $k = p$. If f is p-stable, then the dynamical degree λ_p is just the eigenvalue of largest magnitude for the operator f^* on the (finite dimensional) cohomology group $H^{p,p}(X)$.

A meromorphic map is always N-stable, where $N = \dim_{\mathbf{C}} X$. The condition of 1-stability can be recast in geometric terms that are especially effective when X is a surface. Namely, f fails to be 1-stable if and only if there is a hypersurface $H \subset X$ whose image (proper transform) $f(H)$ under f has codimension larger than 1 but whose image $f^n(H)$ for some $n > 1$ is again a hypersurface.

It was observed in [26] that one can sometimes make a map 1-stable simply by modifying the underlying manifold X.

Theorem 3.1 *Let $f : X \to X$ be a birational map on a compact Kähler surface. Then there is another surface $\hat{X}$ and a birational morphism $\pi : \hat{X} \to X$ that lifts f to an algebraically stable birational map $\hat{f} : \hat{X} \to \hat{X}$.*

The extent to which such a result might hold for arbitrary meromorphic surface maps remains unclear (hence the first condition in Theorem 2.3), and the recent of Favre and Jonsson demonstrates just how subtle this problem might be. Combining a detailed local analysis of the tree structure of the set of all valuations centered at a point with a softer and more global study of an L^2 structure on the Hilbert space obtained by enriching the cohomology of X to include classes associated to any sequence of blowups, they largely succeed [36] in resolving this problem for polynomial maps $f : \mathbf{C}^2 \to \mathbf{C}^2$.

Theorem 3.2 *Let $f : \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial map. Then there exists a rational surface compactifying $\mathbf{C}^2$ such that some iterate of the meromorphic extension of f to X is* 1*-stable.*

Examples of Favre [35] show that the previous two theorems cannot be easily generalized to cover all meromorphic surface maps. Of particular interest is the case of meromorphic maps with small topological degree, since Theorem 2.3, which concerns this class of maps, takes 1-stability as a starting point.

In higher dimensions, almost nothing is known about the stability issue. Some interesting families of examples suggest that the blowing up technique for achieving stability can still be very effective for achieving 1-stability. When $p > 1$, it is not even clear whether there is a good geometric criterion for characterizing p-stability.

It turns out that dynamical degrees can sometimes be used to detect meromorphic mappings with special geometric features. From [26] we have

Theorem 3.3 *Let $f : X \to X$ be a bimeromorphic map of a compact Kähler surface with first dynamical degree $\lambda_2 = 1$. Then, up to bimeromorphic conjugacy, exactly one of the following is true:*

- *f is an automorphism and some iterate of f is isotopic to the identity;*
- *f is an automorphism preserving an elliptic fibration $\pi : X \to \mathbf{P}^1$;*
- *f preserves a rational fibration $\pi : X \to \mathbf{P}^1$.*

This result has been used by Cantat [16] (see also [33]) to analyze the structure of the *cremona group* of all birational self-maps of $\mathbf{P}^2$. He showed, among other things, that this group enjoys a Tits alternative: any finitely generated subgroup contains a free group or an abelian subgroup of finite index.

For more general meromorphic mappings, there are at present only very partial results in the direction of Theorem 3.3. A recent paper [28] by Dinh and Nguyên considers the general situation of $f : X \to X$ is meromorphically semiconjugate to $g : Y \to Y$, establishing necessary relationships between the dynamical degrees of f and g.

Another classification result uses information beyond that of dynamical degrees.

Theorem 3.4 *Suppose that $f : X \to X$ is a 1-stable bimeromorphic map (so that λ_1 is an eigenvalue of f^*) with $\lambda_1 > 1$. Let $\theta \in H^{1,1}(X)$ satisfy $f^*\theta = \lambda_1\theta$. Then f is bimeromorphically conjugate to an automorphism (i.e. a holomorphic birational map) if and only the self-intersection $\theta^2 = 0$.*

This theorem was extended to meromorphic maps with small topological degree in [23], allowing that the holomorphic map conjugate to f might live on a singular surface. Favre [34] further extended the theorem to maps with large topological degree.

4 Examples

There are at present relatively few examples of meromorphic maps in dimension two or larger whose dynamics are understood in detail. It is perhaps likely that our dynamical understanding of general high dimensional mappings will never approach that of rational maps on the Riemann surface. Nevertheless, specific examples have played an important role in the subject so far, and further progress in the subject is likely to depend on finding and understanding more such significant examples.

The polynomial automorphisms of $\mathbf{C}^2$ are probably the best known and best studied examples of multivariable holomorphic maps. In degree two these are the so-called Hénon mappings: $(x, y) \mapsto (y, y^2+c-ax)$. As $a \to 0$, these degenerate to one-dimensional quadratic polynomial maps, and the well-developed theory for complex dynamics of quadratic polynomials therefore furnishes many ideas and analogies for understanding Hénon mappings. Bedford, Lyubich, and Smillie [10–13] laid a general foundation for understanding dynamics of polynomial automorphisms, and Bedford and Smillie went on to consider particular sorts of automorphisms (quasiexpanding maps, complex horseshoes, maps with maximal real entropy) [7–9].

As noted in the introduction, Hubbard was among the first to note the significance of Hénon mappings as complex dynamical system. In collaboration with Papadopol [43] he also considered dynamical systems arising from Newton's method. Partly building on their work, Roland Roeder [46] studied the topology of the Fatou set (i.e. set of normality) in some particular examples related to Newton's method. By contrast, it should be noted that most work so far in higher dimensions has focused on the *complement* of the Fatou set, where the dynamics of a meromorphic mapping are exhibit maximal instability. This is largely because existing tools have seemed useful for analyzing unstable dynamics.

Physics papers, particularly ones from statistical mechanics, have furnished a number of interesting (families of) examples of meromorphic maps. Many of the results in [26] are nicely illustrated by considering various maps in a particular one parameter family of birational mappings that first arose in physics. Bedford and Diller [4, 5] have considered real dynamics of maps in the same family, completely analyzing the case when the real mappings have maximal entropy and more recently looking at parameters where the entropy degenerates to zero. Still more recently, some of the same techniques have been used by Lyubich and Roeder to analyze the dynamics of a non-invertible mapping that describes the zeroes of the partition function for the so-called 'diamond hierarchical lattice'. Lyubich gave two excellent talks on this subject at our BIRS meeting.

While plane birational maps with positive entropy are abundant, there were until very recently few examples of positive entropy *automorphisms* on compact complex surfaces. Bedford and Kim [6] contributed many new examples when they looked at another family of birational maps arising from thermodynamics. Subsequently, McMullen ([44], see also [22]) gave a more synthetic approach to constructing and analyzing these examples.

In addition to examples motivated by physics, there have also been some interesting examples motivated more by considerations from other areas of pure mathematics. For instance, Cantat [15] has given a beautiful recent analysis of some automorphisms that arise in connection with mapping class groups. And, as she described at our BIRS meeting, Sarah Cooke has shown how to use Teichmüller theory to generate many examples (some previously known) of post-critically finite maps on higher dimensional projective space. We mention also a paper of Roeder and Pujals that looks at two dimensional versions of Blaschke products [45].

It is difficult if not impossible to give a completely comprehensive list of the examples that have been studied so far, or to adequately convey the ways in which they have shaped more general work in multivariable complex dynamics. Nevertheless, it is safe to say that there are plenty of other, as yet unconsidered, examples of meromorphic dynamical systems that deserve special attention in their own right and for the light they bring to the subject as a whole.

5 Further considerations

We mention two issues, which among many others, have received recent attention. There have been some attempts at understanding parameter spaces associated to families of meromorphic mappings. Two papers [2, 3] of Bassanelli and Berteloot apply pluripotential theory to study stability issues within families of holomorphic mappings of $\mathbf{P}^k$. The idea here is that the Lyapunov exponents (and various combinations of these) relative to the measure of maximal entropy should vary nicely (i.e pluriharmonically) when dynamics is varying stably within the family.

On a more local front, there has been a lot of interest in understanding the dynamics of a holomorphic mapping near an indifferent fixed point. The general idea here is that in a neighborhood of such a point, the dynamics ought to be those of the unit time flow of a holomorphic vector field. Abate and his coauthors have been particularly active in pursuing this line of inquiry. One can consult [1] for a good recent survey of where the subject stands.

In closing and in this spirit, we would like to point out some further, much longer sources for learning about multivariable complex dynamics. An article by Sibony [47], though a decade old, gives some excellent background and does a good job of conveying the analytic flavor of the subject. More recently, Guedj [39] and Cantat [17] have written two more advanced and comprehensive surveys.

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