Permutation Groups

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1 Overview of the Field

The theory of permutation groups is essentially the theory of symmetry for mathematical and physical systems. It therefore has major impact in diverse areas of mathematics. Twentieth-century permutation group theory focused on the theory of finite primitive permutation groups, and this theory continues to become deeper and more powerful as applications of the finite simple group classification, and group representation theory, lead to astonishingly complete classifications and asymptotic results.

2 Recent Developments and Open Problems

The theory of permutation groups is a classical area of algebra. It originates in the middle of the nineteenth century, with very considerable contributions by most of the major figures in algebra over the last two centuries, including Galois, Mathieu, Jordan, Frobenius, Burnside, Schur and Wielandt. In the last twenty years, the direction of the subject has changed substantially. The classification of finite simple groups has had many applications, many of these through thorough investigation of relevant permutation actions. This in turn led to invigoration of the subject of permutation groups, with interesting new questions arising and techniques developed for tackling them. Interestingly, some topics arose in more than one context, forming new connections. The concept of exceptionality was first suggested by work on covers of curves; it then appeared independently in homogeneous factorizations of graphs, and more recently it has found applications in investigations of line-transitive linear spaces. The concept of derangements in groups (that is, fixed-point-free permutations) and their proportions is classical; it has applications to images of rational points for maps between curves over finite fields, in probabilistic group theory and in investigating convergence rates of random walks on groups. Recently a conjecture of Boston and Shalev on the proportion of derangements in simple group actions has been settled; interestingly, this conjecture fails in the slightly more general case of almost simple groups, through examples of exceptional actions mentioned above. This area continues to be very lively. The topic of fixed point ratios and minimal degrees of elements in permutation groups is classical, going back over 100 years, but there has been significant progress in the last fifteen years both for finite and algebraic groups. It has had applications in arithmetic algebraic geometry, besides leading to significant insights in group theory – a striking example is the solution of Wielandt's conjecture on the characterization of subnormal subgroups. The question of base size of permutation actions is of importance in computational group theory as well as in the study of the graph isomorphism problem. Recent research has thrown much light on the base sizes of actions of almost simple groups in particular. The concept of quasiprimitive permutation groups is also classical, but there has been revived interest in the subject through investigations of groups of graphs and designs. Algebraic graph theory has developed greatly over the last ten to twenty years; there are interesting connections to association schemes and representation theory. Some of these in turn found an application in the study of derangements mentioned above, as well in the study of random walks on groups. Expander graphs is a fairly quickly developing field and there are connections to permutation groups here as well.

We now discuss some of these and related areas and recent developments in more detail.

Permutation groups of small genus.

This area has been motivated by the well-known conjecture of Guralnick and Thompson concerning composition factors of monodromy groups of covers of curves of small genus to the Riemann sphere. The proof of this conjecture has been recently completed, through the work of a number of authors. Work on obtaining explicit lists of composition factors in the very small genus cases continues. These methods and their refinements have led to a complete classification of the monodromy groups of maps from the generic curve of genus $g > 2$, greatly extending classical work of Zariski. There are related conjectures of Guralnick concerning the corresponding situation in positive characteristic, which will require considerable extension of existing methods.

Exceptionality of permutation groups.

The concept of exceptional permutation groups arose in the context of investigations of exceptional polynomials, which arose originally in the work of Dickson, Schur, Davenport, Fried and others. These are polynomials over finite fields which induce permutations on infinitely many finite extension fields. This can be translated into a question relating orbitals of a permutation group and its automorphism group. An answer led to major progress in our knowledge of exceptional polynomials in the work of Fried, Guralnick and Saxl. As a direct consequence, new families of exceptional polynomials were discovered by Lenstra, Zieve and others. Another application appeared in the recent memoir of Guralnick, Müller and Saxl on the rational function analogue of a question of Schur concerning polynomials with integer coefficients which induce permutations on residue fields for infinitely many primes. The solution involved a substantial amount of permutation group theory and algebraic geometry. At about the same time, exceptional permutation groups arose also in the work of Praeger and others on homogeneous factorizations of complete graphs. There is a further recent application in the study of line-transitive linear spaces.

Derangements.

According to a conjecture attributed to Boston and Shalev, there is an absolute lower bound for the proportion of derangements in any action of any simple permutation group. This has been proved recently in an impressive series of papers (and preprints) by Fulman and Guralnick. An important extension, still being investigated, is distribution of derangements in the cosets of the permutation group in its automorphism group. This is connected to the exceptionality condition above. One wants to classify primitive actions in whichmost elements in a coset are not derangements. This would yield information about rational maps and maps between curves over finite fields that are close to being bijective over for arbitrarily large finite fields.

Finite and infinite geometries.

A classification of finite projective planes with automorphism groups primitive on points was obtained by Kantor in the 80's, as a consequence of a classification of primitive permutation groups of odd degree. This was extended to a classification of flag transitive finite linear spaces through the work by Buekenhout and others. Much work has been done recently on line transitive finite linear spaces - there are some interesting problems concerning imprimitivity. Another extension currently under investigation concerns flag transitive finite polygons. One should note that the monumental recent work of Tits and Weiss on Moufang polygons, while concerned with the general problem, does not seem to simplify much in the finite case. However, recent work of Tent on BN-pairs and weak Moufang conditions generalizes many of the classification results to the infinite situation. At the same time, model theory provides techniques to construct (counter-) examples showing that certain results will not generalize to the infinite case.

In this context, also (infinite) split doubly transitive groups should be mentioned. Recent results by Segev, de Medts, Tent, and Weiss seem to make a classification feasible at least in the case of special Moufang sets. This work will also be relevant in the classification of simple groups of finite Morley rank (which again connects the topic to model theory).

Algebraic graph theory.

Successful modern applications of permutation groups in algebraic graph theory date from the late 1980's with proof of the Sims' Conjecture, breakthroughs in the classification of finite distance transitive graphs beginning with the reduction theorem of Praeger, Saxl and Yokoyama, and Weiss's non-existence proof of finite 8-arc transitive graphs of valency greater than 2. These involved use of the simple group classification and built on the theory of finite primitive permutation groups. More recent applications required Praeger's development of the theory of finite quasiprimitive permutation groups. This theory also relies heavily on the finite simple group classification, and has been used successfully to analyse even intransitive finite combinatorial structures such as the Giudici–Li–Praeger theory of locally s-arc transitive graphs. In addition the theory of amalgams and their universal completions forms an important link between infinite graphs and their automorphism groups on the one hand, and classification of finite graphs by their local properties. Much of the geometry associated with the finite simple groups has been elucidated from the study of group amalgams, noting in particular Ivanov's geometric characterisation of the Monster, part of the Ivanov–Spectorov classification of P-geometries and T-geometries. Combining the amalgam approach and the quasiprimitive graph approach is just beginning to pay significant dividends in our understanding of important classes of graphs and group actions. Returning to the finite distance transitive graphs,a related, a slightly more general problem concerns multiplicity free permutation actions. There has been substantial progress towards classification of these actions. Deep character theoretical information on some of these actions has been obtained by Lusztig, Henderson and others. The character tables of the corresponding association schemes have been obtained by Bannai and his coworkers. Some of these actions have been used by Diaconis and others to investigate random walks on groups.

Subgroup structure of finite simple groups.

Theory of primitive permutation groups is closely related to the subgroup structure of finite simple groups and their automorphism groups. There has been impressive progress in this area. For sporadic groups, the answer is almost complete. For alternating groups, the question of maximality was settled in the late eighties through the work of Liebeck, Praeger and Saxl on maximal factorizations of almost simple groups. This reduces the question to classification of maximal subgroups of smaller almost simple groups. For classical groups, Aschbacher's theorem focuses attention on modular representations of almost simple groups; there remain also some questions of non-maximality, currently under investigation. There is some beautiful recent work of Kleshchev and Tiep using very deep ideas in modular representation theory, Hecke algebras and quantum groups. There has been impressive progress in our understanding of subgroup structure of exceptional groups, through the work of Liebeck and Seitz. This is closely linked to subgroup structure of algebraic groups over algebraically closed fields in positive characteristic.

Understanding the maximal subgroups is just the first (but very important) part of understanding the lattice of subgroups. One basic question is a conjecture of Quillen on the contractibility of the subgroup lattice of p-subgroups (the conjecture is that this is the case if and only if there is a nontrivial normal p-subgroup). Another is the question of whether every finite lattice can be embedded in a subgroup lattice of a finite group. This has come up in logic and Banach space theory. There has been considerable progress through the work of Aschbacher, Shareshian, and others.

Infinite permutation groups and model theory

Over the last years there have also been major developments in infinite permutation group theory. One aspect here is the interaction between permutation group theory, combinatorics, model theory, and descriptive set theory, typically in the investigation of first order relational structures with rich automorphism groups. The connections between these fields are seen most clearly for permutation groups on countably infinite sets which are closed (in the topology of pointwise convergence) and oligomorphic (that is, have finitely many orbits on k-tuples for all k); these are exactly the automorphism groups of ω-categorical structures, that is, first order structures determined up to isomorphism (among countable structures) by their first order theory. Here it seems that many new classes of simple groups might be constructed as automorphism groups of such structures. This is particularly interesting when the structures resemble classical objects like projective planes over fields or the like.

Themes of current activity here include the following.

(a) The use of group theoretic means (O'Nan-Scott, Aschbacher's description of maximal subgroups of classical groups, representation theory) to obtain structural results for model-theoretically important classes (totally categorical structures, or much more generally, smoothly approximable structures, finite covers of well-understood structures).

(b) Enumeration and growth rates questions for certain integer sequences associated with oligomorphic groups (e.g. counting the number of orbits on ordered or unordered k-sets – combinatorially well-known sequences frequently arise).

(c) Reconstruction of a first order structure (up to isomorphism, up to having the same orbits on finite sequences, up to 'bi-interpretability') from its automorphism group (typically, presented as an abstract group. Partially successful techniques here include the description of subgroup of the automorphism group of countable index (the 'small index property'), and first order interpretation of the structure in its automorphism group.

(d) Properties which the full symmetric group S on a countable set shares with various other closed oligomorphic groups. We have in mind such properties as: complete description of the normal subgroup structure; uncountable cofinality (that is, the group is not the union of a countable chain of proper subgroups); existence of a conjugacy class which is dense in the automorphism group, or, better, comeagre (or better still, the condition of 'ample homogeneous generic automorphisms'); the Bergman property for a group (a recently investigated property of certain groups G, which state that if G is generated by a subset S, then there is a natural number n such that any element of G is expressible as a word of length at most n in $S \cup S^{-1}$); the small index property. Several of these themes have been linked in recent work of Kechris and Rosendahl motivated partly by descriptive set theory. A closely related issue here is the 'extension property' for a class C of finite relational structures, which stated that if $U \in C$ then U embeds in some $V \in C$ such that every partial isomorphism between substructuresof U extends to an automorphism of V; this condition, proved for graphs by Hrushovski, has connections to the topology on a free group, and to automata theory and issues on the theoretical computer science/finite model theory border.

3 Presentation Highlights

The quality of the participants at the conference and their presentations was quite remarkable. Speakers included Michael Aschbacher, Persi Diaconis and Alex Lubotzky. A number of participants reported that this was one of the best (in fact, the best) conferences they had ever attended.

Aschbacher and Shareshian reported on their joint work about sublattices of the subgroup lattice of a finite group. The basic (still open) question is whether every finite lattice can be embedded as a sublattice of the subgroup lattice of some finite group. This originally came up in the theory of Banach algebras. A natural problem in that context translates precisely to this problem. It is also relevant to problems in logic. It seems pretty clear that this is not the case (and indeed most finite lattices should not be embeddable in a subgroup lattice of a finite group), but even for fairly simple lattices, this cannot be shown yet. Aschbacher has developed a theory which reduces the problem to various questions about almost simple groups (not just about the sublattices of the simple groups but other properties). This is an amazing insight. Using the classification of finite simple groups, Aschbacher and Shareshian hope to settle the problem. A partial result now exists for alternating groups.

Lucchini gave a related talk about simplicial complex associated to the coset poset of classical groups.

Lubotzky reported on recent joint work with Guralnick, Kantor and Kassabov about presentations of finite simple groups. Given the nature of the conference, he focused on the case of alternating and symmetric groups. The main result is that if G is a simple Chevalley group of rank r over a field of size q, then G has a presentation with an absolutely bounded number of generators and relations with the length of the presentation $O(\log n + \log q)$ which is essentially best possible. There is one possible family of exceptions – there are no know bounded presentations of the groups ${}^2G_2(3^{2k+1}$ (it is not expected to be a counterexample). This result applies to alternating and symmetric groups by viewing them as Chevalley groups over the field of size 1 (as suggested by Tits). For symmetric and alternating groups, another result is that they have presentations with 3 generators and 7 relations (or 2 generators and 8 relations). It had not been widely believed that simple groups had bounded presentations. Another related result is getting bounds on second cohomology groups. Here the result is: Let G be a finite group and V an irreducible faithful G-module. Then

$$\dim H^2(G, V) \leq (18.5) \dim V.$$

It is likely the 18.5 can be further reduced to $1/2$. This answers a conjecture of Holt from about 15 years ago.

Liebeck talked about another classical problem. Which triangle groups surject onto groups of Lie type? Recall a triangle group of type (r, s, t) is the group generated by 3 elements x, y, z such that $xyz = 1$ and $x^r = y^s = z^t = 1$. These come up in geometry in many ways. The special case of $(r, s, t) = (2, 3, 7)$ are known as Hurwitz groups and come as automorphism groups of genus g Riemann surfaces with automorphism groups of maximal cardinality – $84(g-1)$ with $g > 1$ (this cannot occur when $g = 2$, the smallest case is $g = 3$ – a surface discovered by Klein).

Diaconis gave a beautiful lecture about Gelfand pairs (these are subgroups of H of G such that the permuation module $\mathbb{C}_H^G$ is multiplicity free (equivalently the endomorphism ring of that module is commutative). There are analogs of these problems for Lie groups as well. This is a classic topic studied by quite a number of people. Recently, there has been considerable interest in another aspect of this – absolutely multiplicity free groups (i.e., every irreducible representation of G restricted to H is multiplicity free or dually every irreducible representation of H induced to G is multiplicity free).

Seitz talked about his recent work with Liebeck on the classification of unipotent conjugacy classes in simple algebraic groups and various applications. This is a topic of central impiortance, which has been widely studied over forty years. While there is extensive literature on the subject, the Liebeck-Seitz works really cleans up the subject. This work has applications to permutation groups – especially involving problems on fixed point ratios. He also gave some nice consequences – for example, showing that every unipotent class is rational in a semisimple algebraic group.

de Medts talked about a generalization of Tits method for proving certain groups are simple. He considered automorphism groups of locally finite trees.

Guiudici and Seress gave talks about classical problems in permutation group theory ($3/2$-transitive groups and orbit equivalent permutation groups).

Van Bon reported on his work with Stroth on locally finite, locally s-arc transitive graphs, and in particular their bound $s \leq 9$ for those graphs with all vertices of valency at least 3. It turns out happily, that an exceptional amalgam their work uncovered had been already addressed by construction of an infinite family of examples in the Giudici–Li–Praeger's global approach, further evidence of the confluence of these too approaches.

Peter Cameron, Peter Neumann and Ben Steinberg gave a trilogy of lectures about various aspects of the exciting new area of synchronising groups and their applications to the theory of finite automata, and to combinatorics.

Reichstein gave a talk about essential dimension and group theory. Essential dimension is a classical subject which has had a resurgence in the past decade (primarily through work of Reichstein).

Segev talked about the abelian root groups conjecture for special Moufang sets, another important area where huge amount of progress has been made recently. Tiep talked about recent work on the irreducibility of representations on subgroups. This is a very important part of the maximal subgroup problem. It involves deep representation as well as recent work on Hecke algebras.

Rob Wilson gave a talk on his new way of describing various of the twisted exceptional groups.

4 Scientific Progress and Outcome of the Meeting

The aim of this workshop was to bring together leading researchers in these related areas as well as those whose research centers on permutation groups. A very successful workshop at Oberwolfach was recently organized on this topic (August 2007) by the same organizers. The range of talks and the results discussed at both conferences were very impressive. The diversity of interests of the participants was rather remarkable. What was even more impressive was the range of discussions and collaborations started and continued during the conferences. It made clear the importance of having such meetings on a regular basis. There were also a fair number of postdocs, women and graduate students.

The Ree groups of type G_2 are the one open case in the theorem about bounded generation for the non-abelian finite simple groups (by Guralnick, Lubotzky, et al) that Alex Lubotzky spoke about in his lecture. Tom De Medts and Richard Weiss, had been working on this case for several months. They gained important new insights from Akos Seress (who has been working on the problem for many years). Even though the problem is still unsolved, it seems no longer out of reach, and the three are now collaborating on it.

De Medts began work with Richard Weiss on Moufang quadrangles of type E_8 mentioned during Weiss's talk. Even though de Medts was already well aware of the important question of trying to find an "exceptional" invariant algebraic structure in the spirit of the classical pseudo-quadratic spaces that arise from Moufang quadrangles of type E_6 and E_7, it was Weiss's lecture that inspired this collaboration. They believe they have indeed discovered a new invariant 32-dimensional algebraic structure, and are working hard to pin it down. This would be a significant step toward solving the larger problem.

Donna Testerman spent a lot of time on a 'book in preparation' with Gunter Malle. She also laid foundations for a joint project with Tim Burness on irreducible embeddings of algebraic groups, to be pursued during a visit by Burness in September.

Several questions that Reichstein asked about representations of p-groups were answered by Rob Wilson and Chris Parker.

Peter Neumann suggested an approach to the solution of a problem on torsion-free uniquely divisible groups raised by Yoav Segev; he was able to answer a question about the number of non-synchronizing degrees of a primitive permutation groups raised by Peter Cameron, and was stimulated by that question to formulate a very strong conjecture about synchronizing semigroups. Csaba Schneider provided solutions to two of questions of Peter Neumann about primitive groups of affine type as synchronizing groups.

We are pleased to report numerous comments made by participants that this was the best workshop they had ever attended. Quoting a participant, 'lively, constructive and informative public discussion [followed] nearly every lecture, and discussion continued long after the formal sessions were over'. This was partly due to the wonderful BIRS facilities. We thank the BIRS for the opportunity to hold such a wonderful meeting at their stunning research facility. Many participants also commented on the BIRS staff and how helpful they were. The organizers in particular are very grateful for all the help received. In particular, the administrator Brenda Williams is singled out for her help both before and during the meeting.

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