Algebraic groups, quadratic forms and related topics

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1 A brief historical introduction

In the early 19th century a young French mathematician E. Galois laid the foundations of abstract algebra by using the symmetries of a polynomial equation to describe the properties of its roots. One of his discoveries was a new type of structure, formed by these symmetries. This structure, now called a "group", is central to much of modern mathematics. The groups that arise in the context of classical Galois theory are finite groups.

Galois died in a duel at the age of 20; his work was not understood or recognized during his lifetime. It took much of the rest of the 19th century for his ideas to be rediscovered, absorbed and applied in other contexts. In the context of differential equations, these ideas were advanced by E. Picard, who, following a suggestion of S. Lie, assigned a Galois group to an ordinary differential equation. This group is no longer finite. It naturally acts on the n-dimensional complex vector space V of holomorphic solutions to the equation. In modern language, the Galois groups that arose in Picard's theory are algebraic subgroup of $\mathrm{GL}(V)$.

This construction was developed into differential Galois theory by J. F. Ritt and E. R. Kolchin in the 1930s and 40s. Their work was a precursor to the modern theory of algebraic groups, founded by A. Borel, C. Chevalley, J.-P. Serre, T. A. Springer, and J. Tits starting in the 1950s. From the modern point of view algebraic groups are algebraic varieties, with group operations given by algebraic morphisms. Linear algebraic groups can be embedded in GL_n for some n, but such an embedding is no longer a part of their intrinsic structure. Borel, Chevalley, Serre, Springer and Tits used algebraic geometry to establish basic structural results in the theory of algebraic groups, such as conjugacy of maximal tori and Borel subgroups, and the classification of simple linear algebraic groups over an algebraically closed field. Considerations in number theory, among others, require the study of algebraic groups over fields that are not necessarily algebraically closed. This more general setting was the primary focus for much of the work discussed in the workshop.

In the 1960s J. Tate and J.-P. Serre developed a theory of Galois cohomology. Serre published his influential lecture notes on this topic in 1964; they have been revised and reprinted several times since then. Galois cohomology can be viewed as an important special case of étale cohomology,

In the 1970s the work of H. Bass, J. Tate and Milnor, established connections among Milnor K-theory, Galois cohomology, and graded Witt rings of quadratic forms. In particular, Milnor asked whether (in modern language) Milnor K-theory modulo 2, is isomorphic to Galois cohomology with $\mathbb{F}_2$ coefficients. A more general question, with 2 replaced by an odd prime, was posed in subsequent work of Bloch and Kato and became known as the Bloch-Kato conjecture.

Since the 1980s there has been rapid progress in the theory of algebraic groups due to the introduction of powerful new methods from algebraic geometry and algebraic topology. This new phase began with the Merkurjev-Suslin theorem which settled a long-standing conjecture in the theory of central simple algebras, using a combination of techniques from algebraic geometry and K-theory. The Merkurjev-Suslin theorem was a starting point of the theory of motivic cohomology constructed by V. Voevodsky. Voevodsky developed a homotopy theory in algebraic geometry similar to that in algebraic topology. He defined a (stable) motivic homotopy category and used it to define new cohomology theories such as motivic cohomology, K-theory and algebraic cobordism. Voevodsky's use of these techniques resulted in the solution of the Milnor conjecture for which he was awarded a Fields Medal in 2002. For a discussion of the history of the Milnor conjecture and some applications, see [46]. The Bloch-Kato conjecture was recently proved by Rost and Voevodsky; see [51, 58, 59, 60, 61, 62, 63].

2 Recent Developments

2.1 Quadratic forms

In the last 20 years there has been a virtual revolution in the theory of quadratic forms. Using motivic methods and Brosnan's Steenrod operations on Chow groups, Merkurjev, Karpenko, Izhboldin, Rost, Vishik and others have made dramatic progress on a number of long-standing open problems in the field. In particular, the possible values of the u-invariant of a field have been shown to include all positive even numbers (by A. Merkurjev, disproving a conjecture of Kaplansky), 9 by O. Izhboldin, and every number of the form $2^n + 1, n \geq 3$ by A. Vishik. (Vishik's result was first announced at our 2006 BIRS workshop.) Another breakthrough was achieved by Karpenko, who described the possible dimensions of anisotropic forms in the nth power of the fundamental ideal I^n in the Witt ring, extending the classical theorem of Arason and Pfister.

In [45] R. Parimala and V. Suresh settled the open question of whether the u-invariant of function fields of p-adic curves is 8 affirmatively if the p-adic field is non-dyadic. Their work relies upon the previous work of D. Saltman on Galois cohomology and on the work of Kato on certain unramified cohomology groups. In a completely different way using patching methods in Galois theory, D. Harbater, J. Hartmann, and D. Krashen reproved this result in [21]. Recently R. Heath-Brown used analytical methods to obtain sufficient conditions for common zeros of systems of quadratic forms over p-adic fields and this result was used by D. Leep to show in particular that the u-invariant of $\mathbb{Q}_p(t_1, \ldots, t_n)$ is 2^{n+2}. This extends the work of [45] and [21] in two significant ways: the transcendence degree need not be 1, and the prime p can be 2. Leep's work is not yet available in the preprint form.

2.2 Algebraic surfaces

An important development in the theory of central simple algebras is the proof by A. J. de Jong, of the long standing period-index conjecture; see [14]. This conjecture asserts that the index of a central simple algebra defined over the function field of a complex surface coincides with its exponent. Previously this was only known in the case where the index of a central simple algebra had the form $2^n \cdot 3^m$ (this earlier result is due to M. Artin and J. Tate). In a subsequent paper de Jong and J. Starr found a new striking solution of the period-index problem by constructing rational points on families of Grassmannians. Yet another geometric approach for the index-period problem was developed by M. Lieblich. Lieblich's approach is based on constructing compactified moduli stacks of Azumaya algebras and studying their properties. Using his geometric methods, M. Lieblich in particular was able to prove a variant of the period-index conjecture for a Brauer group of a field of transcendence degree 2 over $\mathbb{F}_p$. (See [35].)

Similar methods were used by A. J. de Jong, X. He, and J. Starr to establish Serre's conjecture II in the geometric case by showing that every G-torsor over the function field of a complex surface is split. (Here the linear algebraic group G is assumed to be connected and simply connected.) For details, see [15].

The methods they used and their refinements are likely to play an important role in future research on currently open problems in the theory of algebraic groups.

2.3 Cohomological invariants

Many fundamental questions in algebra and number theory are related to the problem of classifying G-torsors and in particular of computing the Galois cohomology set $H^1(k, G)$ of an algebraic group defined over an arbitrary field k. In general the Galois cohomology set $H^1(k, G)$ does not have a group structure. For this reason it is often convenient to have a well-defined functorial map from this set to an abelian group. Such maps, called cohomological invariants have been introduced and studied by J-P. Serre, M. Rost and A. Merkurjev. Among them, the Rost invariant plays a particularly important role. This invariant has been used by researchers in the field for over a decade but the details of its definition and basic properties have not appeared in print until the recent publication of the book [17] by S. Garibaldi, A. Merkurjev and J.-P. Serre.

This book, together with the previous book of M. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol ([29]) have become standard reference sources for current research in algebraic groups.

2.4 Galois theory

Let F be a field containing a primitive p-th root of 1. D. Benson, N. Lemire, J. Mináč and J. Swallow recently gave a complete classification of the non-trivial pro-p-groups G with a maximal closed subgroup which is abelian and of exponent p which are realizable as $G_F/G_E^p[G_E, G_E]$ where G_F is an absolute Galois group and G_E is a subgroup of index p in G_F, was obtained (see [2]).

They also used the Bloch-Kato conjecture to produce new examples of pro-p-groups which cannot be realized as absolute Galois groups.

Consider the p-descending central series $G_F = G_F^{(1)} \supset G_F^{(2)} \supset G_F^{(3)} \supset \ldots$, where $G_F^{[i+1]} = (G_F^{(i)})^p$ $[G_F, G_F^{(i)}]$, and set $G_F^{[i]} = G_F/G_F^{(i)}$.

In the recent paper [11] it is shown that $G_F^{[3]}$ is a Galois-theoretic analogue of Galois cohomology. This group controls Galois cohomology (as a subring of its cohomology ring generated by one-dimensional classes) and $G_F^{[3]}$ can be constructed using Galois cohomology and Bockstein elements in $H^2(G_F^{[2]}, \mathbb{F}_p)$. This is used in obtaining examples of interesting families of pro-p-groups which cannot be realized as absolute Galois groups. The group $G_F^{[3]}$ is interesting. On the one hand, it controls important arithmetic information about the field F, including all non-trivial valuations and orderings. On the other hand, the structure of this pro-p-group appears to be fairly accessible and should be studied further.

2.5 Essential dimension

Essential dimension is a numerical invariant of an algebraic group G, which, informally speaking, measures the complexity of G-torsors over fields. It is is usually denoted by $\mathrm{ed}(G)$.

For finite groups the notion of essential dimension was introduced in 1997 by Buhler and Reichstein in [8, 9] as a natural byproduct of their study of classical questions about simplifying polynomials by Tschirnhaus transformations and algebraic variants of Hilbert's 13th problem. There is also an interesting connection with generic polynomials and inverse Galois theory; see [8], [24, Section 8].

Essential dimension was then defined and studied for (possibly infinite) algebraic groups by Reichstein [49] and Reichstein–Youssin [50]. In this context the theory of essential dimension is a natural extension of the theory of "special groups" initiated by J.-P. Serre in [56]. Over an algebraically closed field k special groups are precisely those of essential dimension 0; these groups were classified by A. Grothendieck [20]. The essential dimension may thus be viewed as a numerical measure of how far a given algebraic group G is from being special. Another such measure is the related invariant of the canonical dimension of G; see [4, 28, 64].

Between 2000 and 2007 the essential dimension has been computed for a number of algebraic groups, using a variety of techniques. One interesting connection is with the notion of cohomological invariant, previously studied by Rost, Serre and others (see Section 2.3): if G has a cohomological invariant of degree d then $\mathrm{ed}(G) \geq d$. Another highly fruitful connection is with the existence of non-toral finite abelian subgroups in G; every such subgroup gives a lower bound on the essential dimension of G; see [50] and [19].

Initially these results were obtained over an algebraically closed base field of characteristic 0, many were then proved under milder assumptions on k; see [3, 13]. On the other hand, even over the field of complex numbers, for many groups G, the problem of computing the essential dimension of G remains wide open. For example, for all but finitely many values of n the projective linear group PGL_n, or the symmetric group S_n are in this category; in this cases the problem of computing $\mathrm{ed}(G)$ is closely related to classical questions in Galois theory and the theory of central simple algebras, respectively. Even for finite cyclic groups $G = \mathbb{Z}/n\mathbb{Z}$ viewed as algebraic groups over the field of rational numbers, the exact value of $\mathrm{ed}(G)$ is not known for most n.

Merkurjev [39] and Berhuy–Favi [3] have further extended the notion of essential dimension to a covariant functor. In this setting the essential dimension of an algebraic group is recovered from its Galois cohomology functor $H^1(*, G)$.

Important developments in this subject have occurred over the past 3 years. The first breakthrough was due to Florence [16] who computed the essential dimension of cyclic p-groups $\mathbb{Z}/p^r\mathbb{Z}$ over a field containing a primitive pth root of unity.

Next came a key idea, due to Brosnan, to study essential dimension in the context of algebraic stacks. To a stack X defined over a field k one associates the functor

$$K \mapsto \text{isomorphism classes of } K\text{-points of } X$$

for any field extension K/k. The essential dimension of X is then defined as the essential dimension of this functor. The class of functors of this form turns out to be broad enough to include virtually all interesting examples, yet geometric enough to be studied by algebro-geometric techniques. There are many important stacks in algebraic geometry, e.g., the moduli stacks of smooth (or stable) curves of genus g or moduli stacks of principly polarized abelian varieties, and it is natural to ask what essential dimensions of these stacks are. These questions are answered in [7].

What is perhaps, more surprising is that stack-theoretic methods have led to strong new lower bounds in the "classical" situation, for some algebraic groups G. Note that in the language of stacks the essential dimension of an algebraic group G is the essential dimension of the classifying stack $\mathrm{B}G$. A key role in establishing this connection is played by the above-mentioned notion of canonical dimension and an incompressibility theorem of Karpenko for p-primary Brauer-Severi varieties [25]. Brosnan, Reichstein and Vistoli [5, 6] recovered Florence's results from this point of view and computed the essential dimension of the spinor group Spin_n for most values of n. Surprisingly, $\mathrm{ed}(\mathrm{Spin}_n)$ increases exponentially in n, while previous lower bounds were linear in n.

Karpenko and Merkurjev [28] refined the techniques of [5] and combined them with new results on Brauer-Severi varieties to give a simple formula for the essential dimension of any finite p-group G over a field containing a primitive pth root of unity. This is a far-reaching extension of the work of Florence [16]. A key ingredient of the proof is an extension of Karpenko's incompressibility theorem to products of p-primary Brauer-Severi varieties.

The Karpenko-Merkurjev theorem and its methods of proof have greatly influenced the research in the area over the past two years. In particular, it led to the solution of several previously open questions about essential dimension; see [42]. There has also been much work on extending Karpenko's Incompressibility Theorem to other classes of varieties, e.g., Hermitian spaces [55] or generalized Brauer-Severi varieties [26]. In [38] the techniques used in the proof of the Karpenko-Merkurjev theorem are further refined to give a general formula for the essential dimension of a larger class of groups, which include twisted p-groups and algebraic tori.

The latter formula was recently used by Merkurjev, in combination with the techniques developed in [40], to give striking new lower bounds on the essential dimension of PGL_n, where $n = p^r$ is a prime power. He shows that $\mathrm{ed}(\mathrm{PGL}_n) \geq (r-1)p^r + 1$. For $r = 2$ this was shown in [40] (and for $r = p = 2$ in [52]). For $r \geq 3$ the best previously known bound was $\mathrm{ed}(\mathrm{PGL}_n) \geq 2r$.

3 Lectures delivered at the workshop

For the purpose of this report we have grouped the 27 lectures presented in the workshop into seven sections as follows. Note that work of the participants is quite interlocked, and some of the talks relate to more than one of these topics.

1. Quadratic forms,
2. Algebraic surfaces,
3. Galois theory and Galois cohomology,
4. Essential dimension,
5. K-theory, Chow groups and Brauer-Severi varieties,

6. Structure of algebraic groups,

7. Representation theory of algebraic groups.

We will now briefly report on the content of each lecture.

3.1 Quadratic forms

Asher Auel: "A Clifford invariant for line bundle-valued quadratic forms".

Line bundle-valued quadratic forms on schemes were first implicitly considered in the early 1970s by Geyer, Harder, Knebusch, and Scharlau to study residue theorems, and by Mumford to study theta characteristics. Motivated by the triangular Witt and Grothendieck-Witt groups introduced by Balmer and Walter, and by the investigation of Azumaya algebras with involution on schemes by Knus, Parimala, Sridharan, and Srinivas, the theory of line bundle-valued quadratic forms has only recently taken on its own significance.

A line bundle-valued quadratic form $(\mathcal{E}, q, \mathcal{L})$ on a scheme X (where 2 is invertible) is the data of a locally free $\mathcal{O}_X$-module (vector bundle) $\mathcal{E}$ of finite rank, an invertible $\mathcal{O}_X$-module (line bundle) $\mathcal{L}$, and a symmetric $\mathcal{O}_X$-module morphism $q : \mathcal{E} \otimes \mathcal{E} \to \mathcal{L}$. A classical quadratic form on X is a line bundle-valued quadratic form with values in the trivial line bundle $\mathcal{O}_X$. A line bundle-valued quadratic form may be thought of as a family, over the points of X, of vector spaces with a quadratic forms taking values in a one dimensional vector space without a fixed choice of basis. Important examples arise from the middle exterior powers of cotangent bundles of smooth varieties of dimension divisible by 4.

The first natural cohomological invariant of a quadratic form, the discriminant, generalizes to line-bundle valued quadratic forms of even rank by the work of Parimala and Sridharan. This current work concerns the construction of the second natural invariant, the Clifford invariant, to line bundle-valued quadratic forms. The classical construction of the Clifford invariant (of an even rank quadratic form) as the Brauer class of the full Clifford algebra does not generalize to line bundle-valued quadratic forms. By the work of Bichsel and Knus, there is no full Clifford algebra of a line bundle-valued form with values in a nonsquare line bundle. This can be interpreted as the nonexistence of a natural "spin" cover of the group of orthogonal similitudes. In its place we have constructed a natural four-fold cover of the group of proper orthogonal similitudes by the even Clifford group. This yields an étale cohomological invariant of line bundle-valued forms of trivial discriminant and rank divisible by 4. This invariant has the novel feature of residing in the 2nd étale cohomology group with μ_4-coefficients $H^2_{\text{ét}}(X, \mu_4)$ and interpolating between the classical Clifford invariant and the 1st Chern class modulo 2 of the value line bundle. In low dimensional cases, this invariant recaptures the classifications of line bundle-valued quadratic forms in terms of reduced norms and pfaffians.

The work of Parimala and Scharlau on the Witt groups of curves over local fields provides examples of 2-torsion Brauer classes that are not represented by the Clifford invariants of quadratic forms. This seems to contradict Merkurjev's theorem over schemes. To the contrary, we conjecture that in the case of curves over local fields, all 2-torsion Brauer classes are represented by Clifford invariants of line bundle-valued quadratic forms.

Eva Bayer-Fluckiger: "Hasse principle for automorphisms of lattices".

An integral lattice is a pair (L, b), where L is a free $\mathbb{Z}$-module of finite rank, and $b\colon L \times L \to \mathbb{Z}$ is a non-degenerate symmetric bilinear form. Over $\mathbb{R}$ we can write b in the diagonal form form $\langle 1, \ldots, 1, -1, \ldots, -1\rangle$. The signature of (L, b) is then defined as (r, s) where r is the number of 1's and s is the number of -1's. We say that b is definite if r or s is 0. Otherwise b is indefinite. (L, b) is called even if $b(x, x) \in 2\mathbb{Z}$, for all $x \in L$.

Fact: $(r - s)$ is divisible by 8.

Assume that $t \in SO(L, b)$ and $r + s = \operatorname{rank}(L)$ is even. Then the characteristic polynomial $f(x) \in \mathbb{Z}[x]$ of t is reciprocal, i.e., $f(x) = x^{\deg f} f(x^{-1})$. Conversely, given a reciprocal polynomial $f(x) \in \mathbb{Z}$, we define:

Definition *(L, b) is an f-lattice if (L, b) is even, unimodular, and there exists $t \in \mathrm{SO}(L, b)$ whose characteristic polynomial equals f.*

Questions. 1) *For which $f \in \mathbb{Z}[x]$ does there exist an f-lattice?* **2)** *For which $f \in \mathbb{Z}[x]$ does there exist an f-lattice with a prescribed signature (r, s)?*

These questions are solved in the definite case. In the indefinite case, D. Gross and C. McMullen provided the necessary conditions on f. These conditions are conjecturally also sufficient if f is irreducible. D. Gross and C. McMullen proved this conjecture if $|f(1)| = |f(-1)| = 1$.

Bayer-Flückiger's main result is the following Hasse Principle for Question 1) above.

Theorem. (Eva Bayer-Fluckiger) *There exists an f-lattice over $\mathbb{Z}$ iff there exists an f-lattice over $\mathbb{Z}_p$.*

Bayer-Flückiger also briefly discussed a similar but somewhat more complicated Hasse Principle for Question 2). She concluded her lecture with several examples.

Detlev Hoffmann: "Differential forms and bilinear forms under field extensions".

The behaviour of algebraic objects such as Galois cohomology groups, Milnor K-groups or quadratic forms under field extensions is an important problem in the study of these objects. For example, a crucial part in the proof of the Milnor conjecture by Orlov-Vishik-Voevodsky relating Milnor K-groups modulo 2 and the graded Witt ring was the determination of the kernel of the map $K_n^M(F)/2 \to K_n^M(E)/2$ between Milnor K-groups modulo 2, where $E = F(q)$ is the function field of a particular type of quadric (given by a certain Pfister neighbor) over a field F of characteristic not 2. In the proof of the Bloch-Kato conjecture, such kernels are again important for field extensions given by function fields of so-called norm varieties as defined by Rost. Another example that has been studied extensively is the behaviour of Witt rings (in characteristic not 2) under field extensions. In general, determining such kernels is very difficult, and only few results are known. For instance, in characteristic not 2, a complete determination of Witt kernels $W(E/F) = \ker(WF \to WE)$ for arbitrary algebraic extensions of degree $[E : F] = n$ is only known for n odd (where the kernel is trivial due to Springer's theorem), for $n = 2$ (easy and well known) and $n = 4$ (proved by Sivatski only in 2008).

Here, we consider the case of a field F of characteristic 2 and the Witt ring WF of symmetric bilinear forms over F. It turns out that in this situation, Witt kernels $W(E/F)$ can be determined explicitly for a large class of field extensions going far beyond what is known in the case of characteristic not 2. Let $X = (X_1, \dots, X_n)$ be an n-tuple of variables ($n \geq 1$), and let $g(X) \in F[X]$ be irreducible. The function field $E = F(g)$ is defined to be the quotient field of the integral domain $F[X]/(g)$. If $n = 1$, E is nothing else but a simple algebraic extension. For $n \geq 2$, one obtains function fields of hypersurfaces. We derive a complete and explicit description of $W(E/F)$ in terms of the coefficients of the polynomial $g(X)$. The proof relies heavily on the use of differential forms. More precisely, let F now be a field of positive characteristic $p > 0$ and let $\Omega^n(F)$ denote the Kähler differentials in degree n over F (with respect to the prime field $\mathbb{F}_p$). We compute the kernel $\Omega^n(E/F)$ for function field extensions $E = F(g)$ for arbitrary irreducible $g(X) \in F[X]$. In the case $p = 2$, one can then use a famous theorem by Kato and results by Aravire-Baeza to compute the kernels $I^n/I^{n+1}(E/F)$ for the graded Witt ring, from which the result on $W(E/F)$ follows by some standard arguments.

3.2 Algebraic surfaces

Mark Blunk: "del Pezzo surfaces of degree 6 and derived categories".

M. Blunk's thesis focuses on an explicit description of certain geometrically rational surfaces, *del Pezzo surfaces of degree 6*. He relates del Pezzo surfaces of degree 6 over an arbitrary field F to the following algebraic information: a triple (B, Q, KL), consisting of a separable algebra B of constant rank 9 with center K étale quadratic, a separable algebra Q of constant rank 4 with center L étale cubic, such that B and Q contain $KL := K \otimes_F L$ as a subalgebra, and the corestrictions $\mathrm{cor}_{K/F}(B)$ and $\mathrm{cor}_{L/F}(Q)$ are *split*, i.e, isomorphic to matrix rings. The main result is:

Theorem 0.1. There are bijections, inverse to each other, between the following two sets: The set of isomorphism classes of del Pezzo surfaces of degree 6 over F, and the set of triples (B, Q, KL), modulo the relation: $(B, Q, KL) \sim (B', Q', K'L')$ if and only if there are F-algebra isomorphisms $\phi_B : B \to B'$ and

$\phi_Q : Q \to Q'$ such that ϕ_B and ϕ_Q agree on their restriction to the subalgebra KL. This restriction is then an isomorphism of F-algebras from KL to $K'L'$.

B and Q can be realized as the global endomorphism rings of two vector bundles $\mathcal{I}$ and $\mathcal{J}$ on S. M. Blunk is able to use these vector bundles to give an explicit description of the K-theory of the surface S.

Theorem 0.2. $K_n(S) \cong K_n(F) \oplus K_n(B) \oplus K_n(Q)$, where K_n is the nth Quillen K-functor.

Similarly, the vector bundles $\mathcal{I}$ and $\mathcal{J}$ can be used to relate the derived category of coherent sheaves on S to the derived category of finitely generated modules over the ring $A = \mathrm{End}_{\mathcal{O}_S}(\mathcal{O}_S \oplus \mathcal{I} \oplus \mathcal{J})$, a finite dimensional F-algebra with semi-simple quotient $F \times B \times Q$. In particular, the functor $\mathrm{Hom}(\mathcal{T}, -) : \mathrm{Coh}(S) \to \mod A$ induces a natural equivalence $\mathrm{RHom}(\mathcal{T}, -) : D^b(\mathrm{Coh}(S)) \xrightarrow{\sim} D^b(\mod A)$.

Daniel Krashen: "Patching topologies and local global principles". (Joint work with D. Harbater and J. Hartmann.)

Patching methods were successfully used by D. Harbater in Galois theory. He proved in particular that every finite group is a Galois group of a regular extension of $\mathbb{Q}_p(t)$. Recently some other exciting results in patching theory and its applications to u-invariants in quadratic forms and Brauer groups were obtained by D. Krashen, D. Harbater and J. Hartman. This talk is a preliminary report on the further development of patching theory. Its aim is twofold: to pay a special attention to the relationship between factorization and local-global principles and second, to extend the basic factorization result to the case of retract rational groups, thereby answering a question posed by Colliot-Thélène.

Broadly speaking, for a given field F the patching method is a procedure for constructing new fields F_ξ which will be in certain ways simpler than F and to reduce problems concerning F to problems about various F_ξ. The focus of Krashen's talk was the function field F of a p-adic curve X and different kind of geometric objects associated to it. Using geometric methods Krashen introduced a kind of "completions" F_ξ of F and using patching technique he talked about local-global principles for Brauer groups, quadratic forms, homogeneous varieties and etc. The details, references and some examples are in [31].

Raman Parimala: "Degree three Galois cohomology of function fields of surfaces". (Joint work with V. Suresh.)

A few years ago Parimala and Suresh proved a long standing conjecture that the u-invariant of the function field of a curve over a p-adic field where $p \neq 2$ is 8. Their proof heavily depends on properties of degree three Galois cohomology of function fields of curves. In her talk Parimala discussed local-global principle for degree three Galois cohomology of function fields of surfaces.

Theorem. (Parimala and Suresh). Let X is a regular 2-dimensional, excellent integral scheme, $F = F(X)$, $l \in \mathcal{O}_x^*$, $\mu_l \in F$. Let Ω be the set of discrete valuations of F associated to the points of $x \in X^1$ of codimension 1. Suppose $H^3_{nr}(F(X), \mu_l) = 0$, and $H^2_{nr}(k(x), \mu_l) = 0, \forall x \in X^1$. Then an element $\xi \in H^3(F, \mu_l)$ is divisible by $\alpha = (a)(b) \in H^2(F, \mu_l)$ if and only if it is divisible locally for all $v \in \Omega$.

Parimala also explained several applications of this local-global principle in computing u-invariant, studying properties of a conic fibration $Y \to X$ where X is a smooth projective surface over a finite field and describing 0-cycles of varieties over global fields.

David Saltman: "Ramification in bad characteristic".

In the past, Saltman obtained important results on central simple algebras over function fields of p-adic curves, by carefully examining ramifications. These results were used by R. Parimala and V. Suresh in showing that a u-invariant over a non-dyadic p-adic function field, is 8, and they are also clearly of independent interest. One particularly interesting motivation is the long-standing problem of whether each division algebra of degree p is cyclic.

In his talk, D. Saltman examined the most difficult case of mixed characteristic. Let S be a nonsingular surface with a field of fractions $K = F(S)$. For every curve $C \subset S$ consider the stalk $\mathcal{O}_{s,c}$. Then
$\mathrm{Br}(S) = \cap_{C \subset S} \mathrm{Br}\,(\mathcal{O}_{S,c}) \leq \mathrm{Br}(F(S))$.

The key problem is to describe ways to split a central simple algebra α over $F(S)$ where the order of α in the Brauer group is not a unit in the residue field. In order to focus on the main difficulty, the following case investigated by K. Kato, was discussed.

$K =$ a fraction field of R, R is a discrete valuation ring, char $K = 0$, char $\bar{R} = p \neq 0, K$ is complete, $[\bar{R} : \bar{R}^p] = p, e = v(p) =$ ramification index, $N = \frac{ep}{p-1}$, K contains a primitive pth-root of unity. (Hence $(p-1)/e)$ $\mathrm{br}(K) =$ elements in the Brauer group of K of order p.

The filtration on units induces filtration on $\mathrm{br}(K)$: $\mathrm{br}(K)_0 \supseteq \mathrm{br}(K)_1 \supseteq \cdots \supseteq \mathrm{br}(K)_{N+1} = \{0\}$. Kato proved: **a**) $\mathrm{br}(K)_0/\mathrm{br}(K)_1 = k^*/k^{*p}$ $(k = \bar{R} =$ residue field of $R)$, **b**) $\mathrm{br}(K)_i/\mathrm{br}(K)_{i+1} = \Omega_k$ if $p \nmid i$, **c)** $\mathrm{br}(K)_i/\mathrm{br}(K)_{i+1} = k^+/k^{+p}$ if $p \mid i$, **d)** $\mathrm{br}(K)_n \cong H^1(k, \mathbb{Q}/\mathbb{Z})$.

Moreover, every element in **a), b), c)** can be represented by a single symbol and can be split by a pth-root of some unit. Saltman discussed several ideas, conjectures and examples in this setting.

Jason Starr: "Rational simple connectedness and Serre's "Conjecture II" ".

Starr's lecture was devoted to the ideas surrounding his recent work with de Jong on the existence of rational sections to fibrations $X \to B$ over an algebraic surface B and its application to Serre's Conjecture II. Recall that this conjecture says that the Galois cohomology set $H^1(F, G) = \{1\}$ for any semisimple simply connected algebraic group G defined over a perfect field F of cohomological dimension at most 2. Equivalently, the question is whether every G-torsor over Spec (F) is trivial. For history and details we refer to the survey [18].

The proof of the geometric case of Serre's Conjecture II (i.e. when F is the function field of a surface over an algebraically closed field k) in [15] is an outgrowth of a project of finding an algebro-geometric analogues of the topological notion of "r-connectedness". The notion of 1-connectedness (also known as rational connectedness) is well understood; the existence of a rational section of $X \to B$ where B is a curve over k and fibers are geometrically connected varieties is a celebrated theorem of Graber, Harris and Starr. The definition of 2-connectedness (also known as rational simple connectedness) is considerably more complicated, but it also implies the existence of a rational section of $\phi : X \to B$ under some natural mild conditions on X, B and ϕ.

In his talk Starr explained how these results are used to complete the proof of Serre's Conjecture II over function fields using P. Gille's inductive strategy.

3.3 Galois Theory and Galois Cohomology

Sanghoon Baek: "Cohomological invariants of simple algebras".

Let A : $\mathsf{Fields}/F \to \mathsf{Sets}$ be a functor. J.-P. Serre defined an *invariant* of A with values in a cohomology theory H (viewed as a functor from Fields/F to Sets) to be a morphism of functors $A \to H$. All the invariants of A with values in H form a group $\mathrm{Inv}(A, H)$. When $A = H^1(-, G)$ for an algebraic group G, we simply write $\mathrm{Inv}(G, H)$ for the group $\mathrm{Inv}(A, H)$. In particular, the cases $A = H^1(-, \mathbf{PGL}_n)$ and $A = H^1(-, \mathbf{GL}_n/\boldsymbol{\mu}_m)$ with m dividing n, i.e., the problems of classifications of invariants of central simple algebras of degree n and central simple algebras of degree n and exponent dividing m, respectively, are still wide open.

Let D be a central simple algebra over a field F. Denote by q_D the quadratic form on D defined by $q_D(x) = \mathrm{Trd}(x^2)$ for $x \in D$, where Trd is the reduced trace form for D. Let $e_n : I^n(F) \to H^n(F)$ be the cohomological invariant for the quadratic form, where $H^n(F) := H^n(F, \mathbb{Z}/2\mathbb{Z})$. Recently, M. Rost, J.-P. Serre and J.-P. Tignol showed that q_D decomposes in the $W(F)$ as the sum of a 2-fold Pfister form q_2 and a 4-fold Pfister form q_4 for $D \in H^1(-, \mathbf{PGL}_4)$ over a base field F such that $\mathrm{char}(F) \neq 2$ and $-1 \in F^{\times 2}$. This provides cohomological invariants e_2 and e_4 given by $D \mapsto e_2(q_2)$ and $D \mapsto e_2(q_4)$ respectively. Another type of cohomological invariants for central simple algebras is from the *divided power* operation: $\gamma_n : K_i(F)/p \to K_{ni}(F)/p$ defined by $\gamma_n(\sum_{j=1}^r \alpha_j) = \sum_{1 \leq j_1 < \cdots < j_n \leq r} \alpha_{j_1} \cdot \ldots \cdot \alpha_{j_n}$, where the α_j are symbols of degree i. In particular, for $p = 2$ and $i = 2$, we have $\gamma_n : \mathrm{Br}_2(F) \simeq k_2(F) \to k_{2n}(F) \simeq H^{2n}(F)$. Restricting the divided powers on the subfunctor $H^1(-, \mathbf{GL}_{2^k}/\boldsymbol{\mu}_2) \subset \mathrm{Br}_2$ we view the γ_n as invariants of $\mathbf{GL}_{2^k}/\boldsymbol{\mu}_2$. Baek discussed his joint work with A. Merkurjev on the invariants of $\mathbf{GL}_{2^k}/\boldsymbol{\mu}_2$ for $1 \leq k \leq 3$. They proved that $\mathrm{Inv}(\mathbf{GL}_n/\boldsymbol{\mu}_2, H)$ is free $H(F)$-module with basis $\{1 = \gamma_0, \gamma_1, \ldots, \gamma_k\}$ for $n = 2^k$ and $1 \leq k \leq 3$. Furthermore, for any $D \in H^1(-, \mathbf{GL}_8/\boldsymbol{\mu}_2)$, the form q_D is a 6-fold Pfister form such that $e_6(q_D) = \gamma_6(D)$. As a consequence, we get $6 \leq \mathrm{ed}(\mathbf{GL}_8/\boldsymbol{\mu}_2) \leq 8$. At the end of the lecture Baek showed that similar result holds for an upper bound of $\mathrm{ed}(\mathbf{GL}_8/\boldsymbol{\mu}_2)$ if the base field F is of characteristic 2.

Skip Garibaldi: "Applications of the degree 5 invariant of E_8".

Recently Nikita Semenov discovered a new degree 5 cohomological invariant for E_8-torsors. (Invariants of torsors appeared also in the talk by Sanghoon Baek.) The construction of this invariant used motives (the technology underlying the proof of the Bloch-Kato conjecture), and unfortunately this does not give an explicit formula for the invariant. S. Garibaldi spoke on his joint work with Semenov where they produce a formula for the invariant for those torsors that appear in Tits construction and gave several applications. Specifically, they constructed new cohomological invariants for curtain groups of type E_7; constructed new examples of anisotropic groups of E_8; constructed new cohomological invariants of Spin_{16}-torsors; computed the essential dimension of the kernel of the Rost invariant on Spin_{16} (connecting his talk with other talks on essential dimension by A. Meyer, R. Lötscher, and M. MacDonald), and used the invariant of E_8-torsors to give concrete criteria for embedding certain finite simple groups in the split form of E_8, filling in a question mark from a 1998 note by Serre.

Arturo Pianzola: "Applications of Galois cohomology to infinite dimensional Lie theory". (based on joint projects with B. Allison, S. Berman, P. Gille, V. Kac, and M. Lau.)

Pianzola's talk focused on surprising connections between of non-abelian Galois cohomology of Laurent polynomial rings and extended affine Lie algebras (a class of infinite dimensional Lie algebras which, as rough approximations, can be thought off as higher nullity analogues of the affine Kac-Moody Lie algebras).

Though the algebras in question are in general infinite dimensional over the given base field (say the complex numbers), they can be thought as being finite *provided that the base field is now replaced by a ring* (in this case the centroid of the algebras, which turns out to be a Laurent polynomial ring). This leads us to the theory of reductive group schemes as developed by M. Demazure and A. Grothendieck. Once this point of view is taken, the language of torsors arise naturally. This novel geometrical approach has lead to unexpected interplays between infinite dimensional Lie theory and the theory of algebraic groups, such as the work of Raghunathan and Ramanathan on torsors over the affine space, isotriviality questions for Laurent polynomial rings, Azumaya algebras, and Serre's Conjecture I and II.

This new language is so flexible and powerful that can be adapted also to the study of Differential Conformal Superalgebras. This involves, at the very least, rewriting the descent formalism for the case when a base scheme is replaced by a *differential* scheme. Concrete application have already been found that relate to the classification of the "affine" N-conformal superalgebras, and work of Schwimmer and Seiberg.

Andrew Schultz: "The first Galois cohomology group as a $\mathrm{Gal}(E/F)$-module, and applications". (Joint with Ján Mináč and John Swallow.)

The talks of A. Schultz and J. Swallow are surveys of recent results on the Galois module structure of Galois cohomology and their applications to Galois theory. A. Schultz began by considering how certain Galois embedding problems related to Kummer theory could be interpreted in terms of the Galois structure of $E^\times/E^{\times p}$, where $E^\times$ represents the multiplicative group of E. Investigations into the structure of this module began with work of Borevič and Faddeev in the case that E is a local field. Schultz presented the following result for extensions satisfying $\mathrm{Gal}(E/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$; in the following result, E_i is the extension of degree p^i of F within E/F.

Theorem. If $p > 2$ and $\xi_p \in F$, and if $\mathrm{Gal}(E/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$, then $E^\times/E^{\times p} \simeq X \oplus Y_0 \oplus Y_1 \oplus \cdots \oplus Y_n$ where each Y_i is a free $\mathbb{F}_p[\mathrm{Gal}(E_i/F)]$-module, and X is cyclic module of dimension $p^{i(E/F)} + 1$. The invariant $i(E/F)$ comes from the set $\{-\infty, 0, 1, \cdots, n-1\}$, where $p^{-\infty}$ is defined to be 0.

One can interpret $i(E/F)$ in terms of embedding problems: $i(E/F) = -\infty$ if E/F can be embedded in a cyclic, $\mathbb{Z}/p^{n+1}\mathbb{Z}$ extension E'/F, and otherwise $i(E/F)$ represents the smallest number i such that E/E_{i+1} can be embedded in a cyclic $\mathbb{Z}/p^{n-i}\mathbb{Z}$-extension E'/F.

This result has analogues in the cases $p = 2$ as well as when $\xi_p \notin E$, but they weren't discussed for expository reasons. The full results are in [44].

Schultz explained how this theorem could be used to show that the appearance of certain Galois groups over F can force the appearance of other Galois groups over F, corollaries in the vein of so-called automatic realization results. The expectation is that the Galois structure of $E^\times/E^{\times p}$ will be used in arithmetic and geometric constructions beyond Galois theory, much in the same way that the structure of $E^\times/E^{\times 2}$ can be used to understand quadratic forms when E is a quadratic extension of F.

John Swallow: "Galois cohomology groups as Galois modules, and applications". (Joint work with D. Benson, J. Labute, N. Lemire and J. Mináč.)

Let p be prime and ξ_p a primitive pth root of unity. Let $k_m F$ denote the reduced Milnor K-theory of the field F modulo p, and let $H^m(F)$ denote the cohomology group $H^m(G_F, \mathbb{F}_p)$. The Bloch-Kato conjecture (now the Rost-Voevodsky theorem) tells us that the norm residue map $k_m F \to H^m F$ is an isomorphism. The purpose of this talk was to explicitly interpret this powerful theorem in terms of structural properties of absolute Galois groups.

To begin, Swallow showed how this theorem forced a stratification in the Galois module structure of certain Galois cohomology groups. Let U be an open normal subgroup of index p in G_F. We write $G := G_F/U$, with E the fixed field of U. Kummer theory shows that $E = F(\sqrt[p]{a})$ for some $a \in F^\times$. We then have the following

Theorem. [LeMS] When viewed as a $\mathbb{F}_p[G]$-module, $H^m E$ is a direct sum of indecomposable submodules of dimensions 1, 2 and p.

Indeed, one can be quite explicit in this decomposition. For instance, one can give the multiplicities of each summand type in terms of arithmetic information related to (a), (ξ_p) and quotients of the filtration $H^{m-1}F \supseteq \mathrm{ann}\{a, \xi_p\} \supseteq \mathrm{ann}(a)$, where $\mathrm{ann}(\cdot)$ denotes the annihilator of the given cohomology class.

The power of this result is exhibited by its applications. For instance, one can use this result to give certain "hereditary" properties of Galois cohomology.

The Bloch-Kato conjecture also allows us to translate certain questions about pro-p groups to the context of Galois cohomology; indeed, the inflation map gives an isomorphism $\inf : H^i(G_F(p), \mathbb{F}_p) \to H^i(G_F, \mathbb{F}_p)$ for all $i \in \mathbb{N}$, where $G_F(p)$ is the maximal pro-p quotient of G_F. One can then ask how standard cohomological properties are translated in terms of these Galois modules. The computed module structure then gives

Theorem. [LLMS] The cohomological dimension of $G_F(p)$ is at most n if and only if $\mathrm{cor} : H^n E \to H^n F$ is surjective for all E/F cyclic of degree p.

One can also give an interesting generalization of Schreier's formula using the Galois module structure of Galois cohomology. Recall that Schreier's formula tells us that if the cohomological dimension of a pro-p group G is 1, then for all open subgroups H in G, $h_1(H) = 1 + [G : H](h_1(G) - 1)$, where $\dim_{\mathbb{F}_p} H^i(H, \mathbb{F}_p) = h_i(H)$. Using the stratified decomposition of Galois cohomology in our case, we have

Theorem. Suppose $h_n(G_F) < \infty$ and that $\mathrm{cor} : H^n E \to H^n F$ is surjective. Then $h_n(G_E) = a_{n-1}(E/F) + p(h_n G_F - a_{n-1}(E/F))$, where $a_{n-1}(E/F) = \dim_{\mathbb{F}_p} \frac{H^{n-1}F}{\mathrm{ann}(a)}$.

One can further develop a formula for the partial Euler-Poincaré characteristic and classify certain small quotients of absolute Galois groups. For details see [34] and [2].

3.4 Essential Dimension

Alexander Duncan: "Groups of essential dimension 2".

Let G be a finite group and $\mathbb{C}$ be the field of complex numbers. A theorem of Buhler and Reichstein asserts that $\mathrm{ed}_{\mathbb{C}}(G) = 1$ if and only if G is cyclic or dihedral. The proof is based on the fact that the only rational complex curve is $\mathbb{P}^1$.

Duncan spoke on his recent classification of finite groups of essential dimenson 2 over $\mathbb{C}$. Here the underlying geometry is considerably more difficult. The minimal rational surfaces with the action of a finite group G were classified by F. Enriques, Yu. Manin, and V. A. Iskovskikh, but this classification is rather involved, and it is not always clear which surfaces occur for a given G.

The starting point of Duncan's work was a recent classification of finite subgroups of the 2-dimensional Cremona group by I. Dolgachev and Iskovskikh, and the following recent results on the essential dimension of finite groups.

- (H.-P. Kraft, R. Lötscher and G. W. Schwarz) Let G be a finite group whose center is non-trivial. Then $\mathrm{ed}(G) = 2$ if and only if G embeds into $\mathrm{GL}_2(\mathbb{C})$.

- (N. Karpenko and A. Merkuriev) Let G be a finite p-group. Then $\mathrm{ed}_{\mathbb{C}}(G)$ is the minimal value of $\dim(\rho)$, where ρ ranges over the faithful complex linear representations of G.

Duncan's main result is the following theorem.

Theorem Let G be a finite group. Then $\mathrm{ed}_{\mathbb{C}}(G) \leq 2$ if and only if G is a subgroup of one of the following groups:

1) $T \rtimes D_{12}$ and $|G \cap T|$ is not divisible by 2 or 3,

2) $T \rtimes D_8$ and $|G \cap T|$ not divisible by 2,

3) & 4) $T \rtimes S_3$ and $|G \cap T|$ is not divisible by 3, (there are two such group up to isomorphism),

5) The general linear group $\mathrm{GL}_2(\mathbb{C})$,

6) The finite projective linear group $\mathrm{PSL}_2(\mathbb{F}_7)$;

7) The symmetric group S_5.

The most intricate parts of Duncan's proof are based on the results he obtained about the Cox ring of a toric variety with a finite group action. These intermediate results are of independent interest.

Roland Lötscher: "A multihomogenization technique for the study of essential dimension of algebraic groups".

Let k be a field, and G be a finite group. A rational covariant of G is the G-equivariant map $\varphi : V \dashrightarrow W$, where V and W are G-modules. φ is called *generically free* if $\overline{\varphi(\mathbb{V})}$ is generically free. $\dim \varphi :=$ dimension of $\overline{\varphi(\mathbb{V})}$. The essential dimension $\mathrm{ed}_k\, G$ can be expressed in terms of rational covariants: $\mathrm{ed}_k(G) = \min\{\dim \varphi \;\big|\; \varphi$ is a generically free covariant of G over $k\} - \dim G$. The related notion of *covariant dimension* $\mathrm{covdim}_k(G)$ defined in a similar manner, using regular, rather than rational covariants. It is easy to see that

$$\mathrm{ed}_k(G) \leq \mathrm{covdim}_k(G) \leq \mathrm{ed}_k(G) + 1\,.$$

Reichstein asked for which groups $\mathrm{ed}_k(G) = \mathrm{covdim}_k(G)$.

Lötscher, H. Kraft, and G. W. Schwarz gave a complete answer to this question. Their main result is the following theorem.

Theorem: Let G be a non-trivial finite group. Then $\mathrm{ed}_{\mathbb{C}}(G) = \mathrm{covdim}_{\mathbb{C}}(G)$ if and only if G has a non-trivial center.

The proof relies on a multihomogenization technique pioneered by Florence [16] and further developed by Lötscher, H. Kraft, and G. W. Schwarz. The idea is to replace a faithful covariant $\varphi : V \to W$ by a homogeneous (and more generally, a multihomogeneous) faithful covariant $\varphi_h : V \to W$ such that $\dim(\varphi) \geq \dim(\varphi_h)$.

Lötscher has found other applications of this technique. In particular, it can be used to simplify the proof of the theorem of Karpenko and Merkurjev [28] on the essential dimension of a finite p-group.

Mark MacDonald: "Essential p-dimension of algebraic tori".

MacDonald spoke on his recent joint work with Lötscher, Meyer and Reichstein. The starting point of this project is the following theorem, due to Karpenko and Merkurjev.

Theorem 0: Let G be a finite p-group and k be a field containing a primitive pth root of unity. Then $\mathrm{ed}_k(G;p) = \mathrm{ed}_k(G) = \min \dim(V)$, where the minimum is taken over all faithful k-representations $G \hookrightarrow \mathrm{GL}(V)$.

MacDonald and his collaborators proved similar formulas for for a broader class of algebraic groups G, which includes all twisted p-groups and all algebraic tori. Their main result is as follows.

Theorem 1: Let k be a p-closed field of characteristic $\neq p$. Suppose there exists an exact sequence $1 \to T \to G \to F \to 1$ of algebraic groups over k, where T is a torus and F is a twisted finite p-group. Then: **(a)** $\mathrm{ed}_k(G;p) \geq \min\dim(\rho) - \dim G$, where the minimum is taken over all p-faithful linear representations ρ of G_k over k. **(b)** If G is the direct product of T and F then equality holds in (a). Moreover, $\mathrm{ed}(G) = \mathrm{ed}_k(G;p)$.

Note that for the purpose of computing $\mathrm{ed}(G;p)$, the assumption that k is p-closed is harmless; the value of $\mathrm{ed}(G;p)$ does not change if k is replaced by its p-closure.

If G a direct product of a torus and an abelian p-group, the value of $\mathrm{ed}_k(G;p)$ given in part (b) can be rewritten in terms of the character module $X(G)$. This often renders it computable by standard methods of integral representation theory. In the case of a torus, this results in the following simple formula.

Theorem 2: Let T be an algebraic torus defined over a p-closed field k of characteristic $\neq p$. Suppose the absolute Galois group $\Gamma = \mathrm{Gal}(k)$ acts on the character lattice $X(T)$ via a finite quotient $\overline{\Gamma}$. Then $\mathrm{ed}_k(T) = \mathrm{ed}_k(T;p) = \min\mathrm{rank}(L)$, where the minimum is taken over all exact sequences of $\mathbb{Z}_{(p)}\overline{\Gamma}$-lattices of the form $(0) \to L \to P \to X(T)_{(p)} \to (0)$. with P permutation. Here $X(T)_{(p)}$ stands for $X(T) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

MacDonald outlined a proof Theorems 1 and 2 and discussed several applications. For details, see [38]. Other applications were suggested by workshop participants during the question period.

Aurel Meyer: "A bound on the essential dimension of central simple algebras".

Given a central simple algebra A over a field K, one can ask whether A can be written as $A = A_0 \otimes_{K_0} K$ where A_0 is a central simple algebra over some subfield K_0 of K. In that situation we say that A *descends* to K_0. Let us assume that K contains a base field k, which is assumed to be fixed throughout. The *essential dimension* of A, denoted $\mathrm{ed}(A)$, is the minimal transcendence degree over k of a field $k \subset K_0 \subset K$ such that A descends to K_0. It can be thought of as "the minimal number of independent parameters" required to define A.

For a prime number p, the related notion of essential dimension at p of an algebra A/K is defined as $\mathrm{ed}(A;p) = \min\, \mathrm{ed}(A_{K'})$, where K'/K runs over all finite field extensions of degree prime to p. We also define $\mathrm{ed}(\mathrm{PGL}_n) := \mathbf{max}\,\{\, \mathrm{ed}(A)\,\}$, and $\mathrm{ed}(\mathrm{PGL}_n;p) := \mathbf{max}\,\{\, \mathrm{ed}(A;p)\,\}$, where the maximum is taken over all fields K/k and over all central simple K-algebras A of degree n. The appearance of PGL_n in the symbols $\mathrm{ed}(\mathrm{PGL}_n)$ and $\mathrm{ed}(\mathrm{PGL}_n;p)$ has to do with the fact that central simple algebras of degree n are in a natural bijective correspondence with PGL_n-torsors.

The problem of computing $\mathrm{ed}(\mathrm{PGL}_n)$ was first raised by C. Procesi in the 1960s in the context of his (and S. Amitsur's) pioneering work on universal division algebras. Procesi showed (using different terminology) that in fact, $\mathrm{ed}(\mathrm{PGL}_n) \leq n^2$; see [48, Theorem 2.1].

Meyer talked about the following new upper bounds on the essential p-dimension of the projective linear group PGL_{p^r}: $\mathrm{ed}(\mathrm{PGL}_n;p) \leq 2\frac{n^2}{p^2} - n + 1$. L. H. Rowen and D. J. Saltman [53] showed that if $s \geq 2$ then there is a finite field extension K'/K of degree prime to p, such that $A' := A \otimes_K K'$ contains a field F, Galois over K' with $\mathrm{Gal}(F/K') \simeq \mathbb{Z}/p \times \mathbb{Z}/p$. The above bound is thus a consequence of the following theorem.

Theorem: Let A/K be a central simple algebra of degree n. Suppose A contains a field F, Galois over K and $\mathrm{Gal}(F/K)$ can be generated by $r \geq 1$ elements. If $[F:K] = n$ then we further assume that $r \geq 2$. Then $\mathrm{ed}(A) \leq r\frac{n^2}{[F:K]} - n + 1$.

Meyer explained how to prove this theorem. The construction of a suitable subalgebra A_0 is based on the theory of $\mathrm{Gal}(F/K)$-lattices. For details, see [43].

3.5 K-theory, Chow Groups and Brauer-Severi Varieties

Mikhail Borovoi: "Extended Picard complexes and homogeneous spaces". (Joint work with Joost van Hamel.)

Inspired by a result of Kottwitz, for a smooth algebraic variety X over a field k of characteristic 0, Borovoi and van Hamel introduce a certain complex of Galois modules $\mathrm{UPic}(X)$, which they call the extended Picard complex of X. From $\mathrm{UPic}(X)$ one can compute the Picard group $\mathrm{Pic}(X)$ and the algebraic Brauer group $\mathrm{Br}_a(X)$. Borovoi and van Hamel compute $\mathrm{UPic}(G)$ (up to an isomorphism in the derived category), where G is a connected linear algebraic group over k. Moreover, they compute $\mathrm{UPic}(X)$ (again up to an isomorphism in the derived category) where X is a homogeneous space of a linear algebraic group over k (they do not assume that X has a k-point). This permits them to compute $\mathrm{Br}_a(X)$ for such X. In the course of the proof they consider the equivariant Picard group $\mathrm{Pic}_G(X)$, where now k is an algebraically closed field of characteristic 0 and X is any integral variety over k with any action of a connected k-group G. They compute $\mathrm{Pic}_G(X)$ in terms of divisors and rational functions (on X and on $X \times_k G$).

Baptiste Calmès: "Invariants, torsion indices and oriented cohomologies of flag varieties". (Joint work with Viktor Petrov and Kirill Zainoulline.)

After the work of M. Levine and F. Morel on algebraic cobordism, it is a natural program to try and lift the calculations from specifically-oriented cohomology theories such as Chow groups, the Grothendieck group K, and connective K-theory, to any oriented cohomology h in the sense of M. Levine and F. Morel. In joint work with V. Petrov and K. Zainouilline, B. Calmès succeeded in adapting Demazure's 1973 calculation of the Chow ring of G/B, where G is a semisimple, simply connected linear algebraic group G over a field k, and B is its Borel subgroup to such a calculation of $h^*(G/B)$ where h^* is any oriented cohomology.

As an application, they prove a generalization to all oriented cohomology theories, Borel's description of the singular cohomology of complete flags of type A_n in terms of symmetric polynomials. Also they provide an algorithm to compute the ring structure of the algebraic cobordism of G/B.

Nikita Karpenko: "Incompressibility of quadratic Weil transfer of Severi-Brauer varieties".

Recall that if X is a smooth complete irreducible variety X/F, then X is incompressible if any rational map $X \dashrightarrow X$ is dominant. Equivalently, canonical dimension of $X = \dim X$. Let K/F be a separable quadratic extension, and let $\mathcal{D}/K$ be a 2-primary division algebra such that $N(\mathcal{D}) =$ a corestriction of $\mathcal{D}$ from K down to F is Brauer-trivial. Let $\mathcal{SB}(\mathcal{D})$ be the Severi-Brauer variety of $\mathcal{D}$ and $R(\mathcal{SB}(\mathcal{D}))$ be its Weil transfer.

Theorem. *Then the variety $R(\mathcal{SB}(\mathcal{D}))$ is 2-indecomposable (hence 2-incompressible).*

One can consider generalized Severi-Brauer varieties $\mathcal{SB}_{2^i}(\mathcal{D}), i = 0, 1, \ldots, n$ (where the degree of $\mathcal{D}$ is 2^n). One can still prove that $R\mathcal{SB}_{2^i}(\mathcal{D})$ is 2-incompressible. The proof uses some very interesting motivic decompositions of the motives of these varieties.

It is known that a non-hyperbolic orthogonal involution on a central simple algebra A remains non-hyperbolic after passing to the function field of SB(A). J.-P. Tignol recently observed that the same is true for unitary involutions on algebras of exponent 2. Karpenko's work was motivated by trying to extend this observation to unitary involutions on algebras of arbitrary exponent.

Max-Albert Knus: "Severi-Brauer varieties over the field with one element". (Joint work with Jean-Pierre Tignol.)

A field $\mathbb{F}_1$ with one element may look humorous, but in fact it has recently attracted considerable attention and inspiration.

The idea of a field $\mathbb{F}_1$ first showed up in a paper published by M. J. Tits in 1957. In that paper Tits associated geometries to Dynkin diagrams. Let D be a Dynkin diagram. Let $G_F(D)$ be a Chevalley group over a field F attached to D and let $W(D)$ be a corresponding Weyl group. Tits showed that there exist unique geometries $\Gamma_F(D)$ and $\Gamma_w(D)$ such that the automorphism groups of the geometries are resp. $G_F(D)$ and $W(D)$. Tits called the geometries $\Gamma_w(D)$ attached to Weyl groups, *geometries over the field* $\mathbb{F}_1$ *of characteristic* 1.

Example. Geometry of type A_{n-1} over $\mathbb{F}_1$. $\mathbb{P}^{n-1}\mathbb{F}_1 \stackrel{\text{def}}{=}$ an n-element set X. The projective geometry of dimension $n-1$ over $\mathbb{F}_1$ is $A = S_n$. Observe that $|\mathbb{P}^{n-1}(\mathbb{F}_q)| = \frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1}$. Hence if $q = 1 \Rightarrow |\mathbb{P}^{n-1}(\mathbb{F}_1)| = n$. This explains the "$\mathbb{F}_1$ terminology."

Many properties of usual central simple algebras and central simple algebras with involutions in relation with classical groups have direct analogues over $\mathbb{F}_1$. In particular, one can define exterior powers, Clifford algebras and discriminants in this setting. For example if Γ is an absolute Galois group over F, the étale algebras of dimension n correspond to Γ-projective spaces over $\mathbb{F}_1$ of dimension $n-1$. Some interesting connections between triality and étale algebras were discussed. For details see [30].

Alexander Vishik: "Rationality of integral cycles".

Let k be a field of characteristic 0, Y is a smooth quasiprojective variety over k, F/k is a field extension. Let $\mathrm{Ch}^m(Y) \to \mathrm{Ch}^m(Y_F)$ be the natural map of mth Chow groups of Y and $Y \otimes_k F$. Elements in this image are called k-*rational*. The motivation for this is the calculation of discrete invariants which lead to the construction of fields with a u-invariant equal to $2^s + 1, s \geq 3$.

mod 2 **case.** Q is a smooth projective quadric.

Theorem. *Assume* $\bar{Y} \in \mathrm{Ch}^m(Y_{\bar{k}})/2, m < \frac{\dim Q}{2}$. *Then* $\bar{Y}$ *is* k*-rational* $\Leftrightarrow \bar{Y}$ *is* $k(Q)$*-rational.* (Also it is true in some special cases for $m \geq \frac{\dim Q}{2}$.)

In this talk, A. Vishik discussed the proof of the following theorem.

Theorem. *Assume* $\bar{y} \in \mathrm{Ch}^m(Y_{\bar{k}})$ *and* **(1)** $m < \frac{\dim Q}{2}$, and **(2)** *The first Witt invariant* $i_1(Q) > 1$. *Then* $\bar{y}$ *is* k*-rational* $\Leftrightarrow$ *it is* $k(Q)$*-rational.*

The overall structure of the proof is similar to the mod 2 case, but there are some additional significant additional complications. In particular one uses algebraic cobordism Ω^*, constructions by Levine and Morel, and symmetric cohomological operations on Ω^* introduced by A. Vishik.

3.6 Structure of Algebraic Groups

Philippe Gille: "Algebraic groups with few subgroups". (Joint work with S. Garibaldi.)

If G is a reductive algebraic group G over $\mathbb{C}$, using Dynkin's work one can list all connected reductive subgroups of G. One can also do it over local or global fields. But over "general fields" the situation is significantly more difficult.

In the early 1990s, in his lectures at Collège de France J. Tits showed that "generic" groups of type E_8 have no other connected subgroups than maximal tori. P. Gille's talk was a variation on a theme of Tits' lectures (Gille attended Tits' lectures in the early 1990s being a graduate student). In his talk he gave an alternative proof of the Tits' result based on Totaro's computation of the torsion index of E_8. He also discussed the case of other exceptional groups, in particular the trialitarian case.

Note that in general case the problem of describing reductive subgroups of exceptional groups is still open. Conjecturally all "generic" simple groups of exceptional type have no proper semisimple subgroups.

Alex Ondrus: "Minimal anisotropic groups of higher real rank".

Motivation for A. Ondrus's work is provided by E. Ghys's conjecture which says that if G is a connected, semisimple real Lie group with finite center, rank $G \geq 2$ and Γ is any irreducible lattice in $G(\mathbb{R})$, then Γ has a non-trivial orientation-preserving action on $\mathbb{R}$. The statement is equivalent to saying that Γ has no total order $\leq$ stable by left multiplication. If Γ has such an order then any subgroup also has such an order. Thus to prove Ghys conjecture it suffices to consider almost minimal lattices of higher rank. By the Margulis arithmeticity theorem every such lattice is isomorphic to the group of integer points of a minimal $\mathbb{Q}$-simple algebraic group of higher real rank; hence we arrive to necessity of classification of such groups.

In the isotropic case the classification of such minimal G up to isogeny, was achieved by V. Chernousov, L. Lifschitz and D. W. Morris. They succeeded to do so over any algebraic number field F of higher real rank. A. Ondrus obtained such a classification for anisotropic groups, as follows.

Theorem. *If G is an absolutely simple, minimal anisotropic group over an algebraic number field F, then G is isomorphic to one of the following groups (up to isogeny):*

1) $\mathcal{SU}_3(L,f)$ *for L/F quadratic, f hermitian on L^3 with at least one real place v such that* $L \otimes F_v \cong F_v \times F_v$, *or*

2) $\mathcal{SU}_1(D,\tau)$ *a central division algebra of degree $p \geq 3$ over L/F quadratic with involution of the second kind τ such that either*

- **A**) $L \otimes F_v \cong F_v \times F_v$ *for some real place v, or*
- **B**) $\tau \otimes 1$ *corresponds to a hermitian form of index ≥ 2 over $M_p(\mathbb{C})$ for some real place v.*

3) $SL_1(D), D$ *is a division algebra with* $\deg(D) = p$ *odd.*

Vladimir L. Popov: "Cross-sections, quotients, and representation rings of semisimple algebraic groups".

Let $G \neq \{1\}$ be a connected complex semisimple algebraic group. In 1965 Steinberg proved that if G is simply connected, then there exists a closed irreducible cross-section S of the set of closures of regular conjugacy classes. That is, every such orbit closure intersects S in exactly one point. Equivalently, there exists a regular section of the categorical quotient map $\pi\colon G \to G/\!/G$. This section played an important role in Steinberg's celebrated solution of Serre's Conjecture I.

In a letter to J.-P. Serre, dated January 15, 1969, A. Grothendieck asked whether there exists such a section of π if G is not simply connected. He also asked for which G π has a rational section.

Both problems were solved within the last year. Popov showed that π has a regular section if and only if G is simply connected, and J.-L. Colliot-Thélène, B. Kunyavskiĭ, Popov, and Reichstein, showed that a rational section exists for any G. Moreover, Popov obtained similar results for groups defined over an algebraically closed field of any characteristic. Here, once again, a rational section always exists and a regular section exists if and only if the universal covering isogeny $J : \hat{G} \longrightarrow G$ is bijective on k-points.

Popov also discussed other related questions, such as: What is a minimal generating set of $k[G]^G$? What are the singularities of $G/\!/G$? What is a minimal generating set of the representation ring of G? For details and further references, see [47].

3.7 Representation theory of algebraic groups

Sunil Chebolu: "Freyd's generating hypothesis and the Bloch-Kato conjecture". (Joint work with Jon Carlson, Ido Efrat, and Ján Mináč.)

The generating hypothesis (GH) is a famous conjecture in homotopy theory due to Peter Freyd. It states that a map $\phi\colon X \to Y$ between finite spectra that induces the zero map on stable homotopy groups is null-homotopic. Motivated by this long-standing unsolved problem, the authors formulate and solve its analogue in the stable module category $\mathrm{stmod}(kG)$ of a finite group. It is assumed that characteristic of k is p and p divides $|G|$. Consider the thick subcategory thick (k) generated by k which is the smallest subcategory of $\mathrm{stmod}(kG)$ that is closed under exact triangles and retractions. The main theorem states that the Tate cohomology functor $\widehat{H}^*(G,-)\colon$ thick (k) $\to$ graded k-vector spaces is faithful if and only if the Sylow p-subgroup of G is either C_2 or C_3. Motivated by the general failure of the generating hypothesis for the stable module category, the authors define the ghost number of kG (for a p-group G) to be the smallest non-negative integer l such that the composition of any l ghosts between finite-dimensional kG-modules is trivial in $\mathrm{stmod}(kG)$. They obtain various bounds on these new invariants and compute them in specific groups.

A closely related question is the finite generation problem for Tate cohomology. For which finitely generated kG-modules M is the Tate cohomology $\widehat{H}^*(G,M)$ finitely generated as a module over the Tate cohomology ring $\widehat{H}^*(G,k)$? Motivated by many partial results they proved on finite generation for Tate cohomology, the authors conjecture that if $\widehat{H}^*(G,M)$ is finitely generated over $\widehat{H}^*(G,k)$ then the support variety $V_G(M)$ of M is equal to the entire maximal ideal spectrum $V_G(k)$ of the group cohomology ring.

There was no time to cover the small Galois pro-p-groups which determine entire Galois cohomology and their applications for investigating arithmetic of fields and structure of Galois groups of maximal p-extensions of fields. For details see [11].

Eric Friedlander: "Restrictions to $G(\mathbb{F}_p)$ and $G_{(r)}$ of rational G-modules". (Joint work with J. Carlson, J. Pevtsova and A. Suslin.)

Standard modular representation theory considers as representation spaces, vector spaces over an algebraically closed field k of char$(k) = p > 0, p\big|\,|G|$. Let G be a finite group scheme, $\mathcal{G}$ - a connected reductive algebraic group defined over $\mathbb{F}_p$. $\mathcal{G}(\mathbb{F}_p)$ are points over $\mathbb{F}_p$. Consider rational $\mathcal{G}$-modules M (finite dimensional vector spaces over k).

Frobenius kernel of $\mathcal{G}$: Let $F:\mathcal{G}\longrightarrow\mathcal{G}$ be the Frobenius map. Then set Ker$\{F^r\} = \mathcal{G}_{(r)}\hookrightarrow\mathcal{G}$. Every rational $\mathcal{G}$-module restricts to give a $\mathcal{G}_{(r)}$-module.

Basic Question. *Relate invariants of* $\mathcal{G}(\mathbb{F}_p)$ *and* $\mathcal{G}_{(r)}$ *for various* r.

If M is a rational $\mathcal{G}$-module we can consider Φ_x^*M as a $\mathbb{G}_a$-module. (Here $x^p = 1$ and $\Phi_x:\mathbb{G}_a\longrightarrow\mathcal{G}$ is a "1-parameter subgroup" such that $\Phi_x(1) = x$ and some further restrictions on the image of Φ. (This is work of G. Seitz.)) Hence we obtain a map $\Phi_x^*M\longrightarrow\Phi_x^*M\otimes k[t]$.

$s(M)$ is an important invariant, the least integer such that certain operators indexed by integers $\geq s(M)$ act trivially on M.

Let G be a group scheme and consider Spec $H^\bullet(G,k)$, where $\bullet = *$ if $p = 2$ and $\bullet$ ranges over the even non-negative numbers if $p > 2$. (Note that in both cases $H^\bullet(G,k)$ is commutative.) A well-known theorem of Quillen says that Spec $H^*(\mathcal{G}(\mathbb{F}_r),k) = \mathrm{colim}\, E\otimes k,\ E < \mathcal{G}(\mathbb{F}_r)$, where $E\otimes k$ is an affine space of rank t ($E\cong(\mathbb{Z}/p\mathbb{Z})^t$). A. Suslin, E. Friedlander and C. Bendel showed that Spec $H^*(\mathcal{G}_{(r)},k)\approx V(\mathcal{G}_r)$ where k-points are the 1-parameter subgroups of $\mathcal{G}_{(r)}$. E. Friedlander and J. Pevtsova further found a description of Proj $H^\bullet(G,k)$ using certain equivalence relations on some functions $\alpha: k[t]/t^p\longrightarrow kG$. For a given M one can define the support variety of M. One way to do so is to set $(\Pi\, G)_M = \{[\alpha]:\ \alpha^*M \text{ is not free}\}$.

Theorem (J. Carlson, Z. Lin, D. Nakano) *For a large enough prime p (depending on $\mathcal{G}$) there exists an embedding* $\Pi\,\mathcal{G}(\mathbb{F}_r)\hookrightarrow\Pi\,\mathcal{G}_{(r)}/\mathcal{G}(\mathbb{F}_r)$ *for any* $r\geq 1$ *and if* $p^r\geq S_{\mathbb{F}_r}(M)$*, then* $(\Pi(\mathcal{G}(\mathbb{F}_r))_M\cong\left(\Pi\,\mathcal{G}_{(r)}\right)_M/\mathcal{G}(\mathbb{F}_r)\cap\Pi\,\mathcal{G}(\mathbb{F}_r)$.

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