Commutative Algebra and its Interaction with Algebraic Geometry

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June 10–June 15, 2007

1 Overview of the Field

Commutative Algebra and Algebraic Geometry have been closely connected since the early days of both fields. Many of the concepts in Commutative Algebra have their origins in Geometry, and many of the foundations of Algebraic Geometry are based on Algebraic results.

In this conference we brought together mathematicians who work in the interface of these two fields. Many of them are using Geometric methods to solve questions in Algebra, while others are studying Geometry using methods from Commutative Algebra.

The field is very broad, and our original intent was to concentrate on three main areas. The first deals with problems in Positive and Mixed Characteristic, a topic which has been very active in recent years. The second is in Integral Dependence and Integral Closure, a central topic in Commutative Algebra with many applications to Geometry. The third topic is Secant Varieties and Algebraic Statistics, which presents a new and unexpected application of Geometry to Algebra and other fields.

However, as the preparations for the conference progressed, it became clear that it was better to include a wider range of topics, as there are many important developments that do not fit neatly into any of these three areas. As a result, although the majority of the talks were closely related to the topics that we had originally planned, we also had contributions that can be grouped into other areas, as well as a few that were in special topics. The unifying theme was that all of the speakers discussed problems related to both Commutative Algebra and Algebraic Geometry.

The remainder of this report presents the various areas that were represented and abstracts of the talks. In addition to the three areas mentioned above, there is a section on Intersection Theory and Homological Methods, and more general methods of classical Projective Geometry have been included in the section on Secant Varieties.

2 Problems in positive and mixed characteristic.

The traditional area of study in Algebraic Geometry was spaces defined by equations over the real and complex numbers. However, some problems can be solved by reducing these equations modulo a prime number, and this leads to questions of positive characteristic. A ring R has characteristic p for a prime number p if $pr = 0$ for all r in R. The main advantage this gives is that it allows the use of the Frobenius map, which sends an element r to r^p; it is a ring homomorphism in this case. Most of the theory in positive characteristic involves the Frobenius map in one way or another.

One of the main objects of study in positive characteristic is Hilbert-Kunz multiplicities, which are limits of ordinary multiplicities over iterations of the Frobenius map. A central question is whether Hilbert-Kunz multiplicities are rational, which is addressed in the talk by Paul Monsky.

Hilbert-Kunz functions of singular plane curves

Paul Monsky, Brandeis University

Let G be a homogeneous degree d element of $L[x, y, z]$, where L is the algebraic closure of Z/p. If $q = p^n$, $e_n(G)$ is the colength of the ideal generated by G and the q^{th} powers of x, y and z. Then $e_n(G) = \mu q^2 - R_n$ where μ lies between $3d/4$ and d, and $R_n = O(q)$.

The talk starts with known results about μ and R_n when G is irreducible, and explains ideas in my proof that $R_n =$ (periodic)$q +$ (eventually periodic). It continues with speculation, arising from computer calculations and my work with Teixeira, as to the behavior of R_n when G is reducible. In particular it makes an explicit conjecture as to the value of $e_n(H^j)$ when $p = 2$ and $H = x^3 + y^3 + xyz$. If this conjecture holds the Hilbert-Kunz multiplicity of the 5-variable polynomial $H + uv$ is an irrational element of $Q(\sqrt{7})$—whether Hilbert-Kunz multiplicities can be irrational is an outstanding problem . □

In recent years, another point of intersection of characteristic p methods in Algebra and Geometry has been the study of multiplier ideals. This will be considered further in the next section, but one interesting facet has been a relation with tight closure in positive characteristic. This is examined in the talk of Mircea Mustaţă.

Test ideals vs. multiplier ideals

Mircea Mustaţă, University of Michigan

The talk is a report on joint work with Manuel Blickle, Karen Smith and Ken-ichi Yoshida. It compares the behavior of certain invariants of singularities in characteristic zero (namely, the multiplier ideals and their jumping numbers) with invariants in positive characteristic, the so-called generalized test ideals. The multiplier ideals are by now well-established invariants, defined in terms of divisorial valuations, that can be computed using resolutions of singularities.

On the other hand, the generalized test ideals have been introduced by Hara and Yoshida using techniques inspired by tight closure theory. Results of Hara, Takagi, Yoshida and Watanabe, via reduction mod p, relate the multiplier ideals of a singularity in characteristic zero with the corresponding test ideals in positive characteristic. This connection is quite subtle, revealing deep connections with arithmetic, and there are still very interesting open problems in this area.

It was clear from the beginning that several subtle properties of multiplier ideals, that are proved via vanishing theorems (such as Subadditivity or the Restriction Theorem) have analogues in the context of test ideals, and the proofs are much more elementary. The talk discusses joint work with Yoshida emphasizing the different behavior of test ideals and multiplier ideals: roughly speaking, we show that all the algebraic properties of multiplier ideals that follow from the computation in terms of resolutions, fail for test ideals. A surprising result that also highlights this different behavior: we show that every ideal in a regular F-finite local ring can be written as a test ideal.

The talk also covers results with Blickle and Smith about the behavior of the jumping exponents of test ideals: under certain assumptions, these are all rational and form a discrete set (note the analogy with the jumping numbers of multiplier ideals). □

Shunsuke Takagi also talked about test ideals in positive characteristic and presented some new results on jumping numbers.

Rationality of F-jumping numbers on singular varieties

Shunsuke Takagi, Kyushu University

This talk is based on a joint work with Craig Huneke. Let R be an excellent Noetherian ring of prime characteristic p and I be an ideal which is not contained in any minimal prime ideal of R. Then we say that a real number $t > 0$ is an *F-jumping exponent* of I if $\tau(I^t) \neq \tau(I^{t-\epsilon})$ for all $\epsilon > 0$, where $\tau(I^t)$ is the generalized test ideal of I with exponent t (see [1] for the definition of generalized test ideals). Blickle, Mustaţă and Smith proved that the F-jumping exponents of I are rational and have no accumulation points if R is an F- finite regular ring essentially of finite type over a field or if R is an F-finite regular ring and I is a principal ideal. We generalize their results to the case of strongly F-regular rings. We say that an F-finite reduced ring A of characteristic $p > 0$ is *strongly F-regular* if for every nonzero divisor c of A, there exists $q = p^e$ such that $c^{1/q}A \hookrightarrow A^{1/q}$ splits an A-linear map. The following is our main results. Suppose that R is a strongly F-regular ring of characteristic $p > 0$. Then the set of F-jumping exponents of I have no accumulation points if one of the following conditions holds: (1) $R = \bigoplus_{n \geq 0} R_n$ is a $\mathbb{Q}$-Gorenstein graded ring with R_0 a field and the (Gorenstein) index of R is not divisible by p; (2) R is a $\mathbb{Q}$-Gorenstein ring whose (Gorenstein) index is not divisible by p and I is a principal ideal; (3) $R = \bigoplus_{n \geq 0} R_n$ is a graded ring with R_0 a field and R has finite graded F-representation type (see [2] for the definition of rings with finite graded F-representation type). Every F-jumping exponent of I is a rational number if the condition (1) or (2) holds.

References

[1] Hara, N. and Yoshida, K., *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc. **355** (2003), no.8, 3143–3174.

[2] Smith, K. E. and Van den Bergh, M., *Simplicity of rings of differential operators in prime characteristic*, Proc. London Math. Soc. (3) **75** (1997), no. 1, 32–62. □

Another topic that was presented at the beginning was the main question in mixed characteristic, namely the well known homological conjectures. Work on these conjectures has led to a deep study of the local cohomology of the absolute integral closure of a local integral domain. This was the topic of the talk by Gennady Lyubeznik.

A Property of the Absolute Integral Closure of an Excellent Local Domain in Mixed Characteristic

Gennady Lyubeznik, University of Minnesota

In this talk we presented a proof of the following theorem:

Theorem. Let $(R, \mathfrak{m})$ be a Noetherian local excellent domain of mixed characteristic, residual characteristic $p > 0$ and dimension at least 3. Let $\sqrt{pR}$ (resp. $\sqrt{pR^+}$) be the radical of the principal ideal of R (resp. R^+) generated by p. Set $\bar{R} = R/\sqrt{pR}$ (resp. $\overline{R^+} = R^+/\sqrt{pR^+}$). Then

(i) $H^1_{\mathfrak{m}}(\overline{R^+}) = 0$, and

(ii) every part of a system of parameters $\{a, b\}$ of $\bar{R}$ of length 2 is a regular sequence on $\overline{R^+}$.

This theorem suggests the following.

Question. Let $(R, \mathfrak{m})$ be a Noetherian local excellent domain of mixed characteristic. Is $\overline{R^+}$ a big Cohen-Macaulay $\bar{R}$-algebra, i.e.

(i) is $H^i_{\mathfrak{m}}(\overline{R^+}) = 0$ for all $i < \dim \bar{R}$, and

(ii) is every system of parameters of $\bar{R}$ a regular sequence on $\overline{R^+}$? □

3 Integral Dependence and Integral Closures

The concepts of integral dependence and integral closure are central to Commutative Algebra, and their connections with Algebraic Geometry are, at present, very active fields of research. Their study is closely related to singularity theory, Rees algebras, and multiplier ideals, all of which were discussed at this conference.

The talk by Brian Harbourne (on symbolic powers) is also related to these topics. Harbourne uses, in an imaginative way, previous work on “fat point ideals” , a very geometric idea.

Comparing powers of ideals with their symbolic powers

Brian Harbourne, University of Nebraska–Lincoln

This talk presents work done jointly with Cristiano Bocci (arXiv:0706.3707v1). Consider a homogeneous ideal $I \subset k[\mathbf{P}^N] = R$, where R is the polynomial ring in $N+1$ indeterminates over an algebraically closed field k of arbitrary characteristic. The underlying question is: for which m and r do we have $I^{(m)} \subset I^r$, where $I^{(m)}$ denotes the m-th symbolic power of I? Our approach to this question is to define a quantity, the *resurgence* $\rho(I)$ of I, this being the supremum of all ratios m/r such that I^r does not contain $I^{(m)}$, and to give bounds on $\rho(I)$ in terms of Hilbert function invariants of I. In particular, if, for any homogenous ideal J, $\alpha(J)$ denotes the least degree t such that $J_t \neq 0$ (i.e., $\alpha(J)$ denotes the M-adic order with respect to the maximal homogeneous ideal M of R), we show that $\lim_{m\to\infty} \alpha(I^m)/\alpha(I^{(m)}) \leq \rho(I)$, and, if I defines a 0-dimensional subscheme, we show that $\rho(I) \leq \mathrm{reg}(I)/(\lim_{m\to\infty} \alpha(I^{(m)})/m)$. We obtain these bounds by applying two principles. The first is that if $\alpha(I^r) > \alpha(I^{(m)})$, then I^r does not contain $I^{(m)}$. The second, which holds if I defines a 0-dimensional subscheme, is that if $\mathrm{reg}(I^r) \leq \alpha(I^{(m)})$, then $I^{(m)} \subset I^r$.

As a consequence, among all homogeneous ideals I for which R/I has Krull dimension $N-d+1$ for a given d, we show that the minimum c such that $m \geq cr$ guarantees $I^{(m)} \subset I^r$ is $c = d$. This shows that the well known results of Ein-Lazarsfeld-Smith and Hochster-Huneke are optimal for every dimension and codimension. We also show that $I^{(3)} \subset I^2$ whenever $I = I(S)$ is an ideal of a finite set $S \subset \mathbf{P}^2$ of generic points. This partially answers a still open question of Huneke: if $I = I(S)$ for a finite set S of points in the plane, is it true that $I^{(3)} \subset I^2$? □

3.1 Multiplier ideals and cores

The presentation of Claudia Polini discussed the cores of rings and gives a characterization of certain types of schemes in terms of them. The important interplay between algebra and geometry was also evident in this talk as one of the principal ingredients was the Cayley-Bacharach theorem on finite point sets of projective n-space.

Cayley-Bacharach Schemes and their Cores

Claudia Polini, University of Notre Dame

In the first part of this talk we discuss when the known formulas for cores of ideals are valid in arbitrary characteristic. The core of an ideal I, $\mathrm{core}(I)$, is the intersection of the minimal reductions of I. Being an a priori infinite intersection the core is difficult to compute, and in the last ten years there has been considerable effort to find explicit formulas. There are many reasons to study the core: one is its ties with adjoints and multipliers ideals, another is its connection with Briançon-Skoda type theorems, and last but not least a better understanding of cores could lead to a solution of Kawamata’s conjecture on the non-vanishing of sections of line bundles.

In the second part of the talk we study the annihilators of some graded components of the canonical module of a graded ring. We relate them to cores of powers of homogeneous maximal ideals of standard graded reduced Cohen-Macaulay k-algebras. An application of our results characterizes Cayley-Bacharach schemes in terms of the structure of the core of the maximal ideal of their homogeneous coordinate ring, denoted by $\mathrm{core}(X)$. Recall that a set of s points in $\mathbb{P}^n$ is called a Cayley-Bacharach scheme if every subset of $s-1$ points has the same Hilbert function. In particular, we show that a scheme X is Cayley-Bacharach if and only if $\mathrm{core}(X)$ is a power of the maximal ideal. □

The subject of the talk by Dale Cutkosky is pathological behavior of local cohomology; he shows that this can occur even for Rees algebras.

Rees algebras with non tame local cohomology

Steven Dale Cutkosky, University of Missouri–Columbia

Suppose that R_0 is a local ring, $I \subset R_0$ is an ideal, and $R = R_0[It]$ is the Rees algebra of I. Let $R_+ = ItR$ be the irrelevant ideal of R.

The local cohomology module $H^i_{R_+}(R)$ is tame if either $H^i_{R_+}(R)_j \neq 0$ for all $j \ll 0$ or $H^i_{R+}(R)_j = 0$ for all $j \ll 0$.

Brodmann, Hellus, Lim, Rotthaus and Sega have shown that if $\dim(R_0) \leq 2$, then the local cohomology modules of R are tame.

It has recently been shown by Cutkosky and Herzog that tameness can fail for local cohomology of finitely generated modules over standard graded algebras R with $\dim(R_0) = 3$.

Chardin, Cutkosky, Herzog and Srinivasan have found examples showing that tameness of local cohomology fails for Rees algebras. We describe some of their examples below. In all of the examples, R_0 is normal, generalized Cohen Macaulay, and is essentially of finite type over a field k.

The first example has $\dim(R_0) = 3$, and shows periodic failure of tameness. For $j > 0$, $\dim_k(H^2_{R_+}(R)_{-j})$ is 2 if j is even, and is 0 if j is odd.

The second example shows failure of tameness of local cohomology which is not periodic, and is not even a quasi polynomial (in $-j$) for large j. Specifically, we have for $j > 0$,

$$\dim_k(H^2_{R_+}(R)_{-j}) = \begin{cases} 1 & \text{if } j \equiv 0 \text{ (mod)}(p+1), \\ 1 & \text{if } j = p^t \text{ for some odd } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the characteristic of k is p. We have $p^t \equiv -1$ (mod) $(p+1)$ for all odd $t \geq 0$. A third example is tame, but

$$\lim_{j \to \infty} \frac{\dim_k(H^2_{R_+}(R)_{-j})}{j^3} = 54\sqrt{2},$$

so $\dim_k(H^2_{R_+}(R)_{-j})$ is far from being a quasi polynomial in $-j$ for large j. □

Lawrence Ein gave a talk about current problems in Algebraic Geometry that are related to multiplier ideals.

Inversion of Adjunction

Lawrence Ein, University of Illinois at Chicago

We discuss the numerical invariant minimal log-discrepancy as a measurement of complexity for singularities occuring in higher dimensional birational geometry. The conjecture of Kollár and Shokurov on inversion of adjunction gives a precise comparison between the minimal log-discrepancies of the variety and its hyperplane sections. We discuss how the recent important work of Birkar, Cascini, Hacon and McKernan on the existence of log-minimal models can be applied to the conjecture. We also discuss the approach using the space of arcs introduced by Ein, Mustata and Yasuda. We state a recent theorem of Ein and Mustata on a precise version of the inversion of adjunction for non-local complete intersection Q-Gorenstein varieties. The theorem is also independently proved by Kawakita. Finally, we discussed an application of these results to bounding regularities of Q-Gorenstein varieties in the projective space defined by degree d equations. □

4 Intersection Theory and Homology

One of the areas of Algebraic Geometry that has had a strong impact on Commutative Algebra is Intersection Theory, in particular the homological definition of intersection multiplicities given by Serre in the 1950's. On the one hand, this led to a set of conjectures that have been a central part of the subject for many years. In addition, they started an interest in homological methods that is very active today.

The first two abstracts in this section deal with Intersection Theory. This subject has connections to K-theory as well as Algebraic Geometry. The talk by Srinivas had its origin in attempts to study projective modules using invariants from these fields.

Oriented Intersection Multiplicities

V. Srinivas, Tata Institute

Barge and Morel defined a graded "oriented Chow group" of a smooth variety X over a field, which may be viewed as a quotient of a group of "oriented algebraic cycles" modulo a suitable equivalence relation. More formally, they considered certain complexes of abelian groups $GMW^p_\bullet(X)$, the Gersten-Milnor-Witt complexes, and defined the p-th oriented Chow group to be $H^p(GMW^p_\bullet(X))$. The terms in the complexes are obtained by putting together Milnor K-groups, and powers of the fundamental ideal in the Witt rings, of function fields of subvarieties of X. The complexes are modelled after the Gersten complexes in K-theory.

A precursor was the idea of an oriented 0-cycle, suggested by M. Nori, which led to the Euler class group, considered in works of several mathematicians.

J. Fasel constructed an intersection product on the oriented Chow groups of Barge and Morel, leading to an "oriented Chow ring", which admits a graded ring homomorphism to the "usual" Chow ring of (unoriented) cycles.

In this lecture, I'll give an introduction to this emerging area, and discuss joint work with Fasel, about the idea of intersection multiplicities in this context. This leads to the formulation of an oriented analogue of Serre's vanishing conjecture for intersection multiplicities. I'll also discuss results of Morel, and further joint work with Fasel, on bundles with vanishing Euler class (taking values in the oriented Chow group of points). $\square$

The next results were on the map to the completion of a rings. There had been several questions of whether this was injective on the functors described. The following example of Kurano was also inspired by intersection properties of rings and their behavior under completion.

An example of a local ring R such that $G_0(R)_{\mathbf{Q}} \to G_0(\hat{R})_{\mathbf{Q}}$ is not injective

Kazuhiko Kurano, Meiji University

This is a joint work with V. Srinivas from Tata Institute, India.

For a Noetherian local ring R, let $G_0(R)$ be the Grothendieck group of finitely generated R-modules. Since the completion $R \to \hat{R}$ is injective, it induces the map $G_0(R) \to G_0(\hat{R})$.

In 2001, Kamoi and Kurano proved that the map $G_0(R) \to G_0(\hat{R})$ is injective if R is an excellent local ring that satisfies one of the following 3 conditions, (1) R is henselian, (2) $R = S_{S_+}$, where S is a standard graded ring over a field S_0, (3) R has only an isolated singularity.

However, Hochster gave an example that the map is not injective. In Hochster's example, the ring is non-normal and the kernel is torsion. Recently, Dao gave a new example. In Dao's example, the ring is normal, but the kernel is still torsion.

We constructed an example of a two dimensional (non-normal) ring, that is essentially of finite type over the complex number field such that the map $G_0(R)_{\mathbf{Q}} \to G_0(\hat{R})_{\mathbf{Q}}$ is not injective.

Using the example, we can construct a Noetherian local ring R' such that R' is a Roberts ring, but $\hat{R}'$ is not. $\square$

The talk by Sean Sather-Wagstaff relates the completeness of a ring to vanishing of Ext modules.

Ext-vanishing and ascent of module structures

Sean Sather-Wagstaff, Kent State University

Let $(R, \mathfrak{m}, k)$ be a noetherian local commutative ring. Jensen, Buchweitz and Flenner, and Frankild and

Sather-Wagstaff have shown that the $\mathfrak{m}$-adic completeness property for an R-module M is related to the vanishing of the modules $\operatorname{Ext}^n_R(\widehat{R}, M)$. Here $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R, viewed as and R-module via the natural local ring homomorphism $R \to \widehat{R}$. The results presented in this talk extend these ideas to include other flat local ring homomorphisms, e.g., the map from R to its henselization R^h or any pointed étale neighborhood $R \to S$.

Theorem. (AJF-SSW-RAW, '07) *Let* $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ *be a flat local homomorphism and* M *a finitely generated* R*-module. The following conditions are equivalent:*

(i) *The* R*-module structure on* M *ascends along* φ*.*

(ii) *The evaluation map* $\operatorname{Hom}_R(S, M) \to M$ *is bijective.*

(iii) $\operatorname{Ext}^i_R(S, M)$ *is finitely generated over* R *for each* $i \geq 1$.

(iv) $\operatorname{Ext}^{\geqslant 1}_R(S, M) = 0$.

The speaker presented several consequences of this result and discussed examples showing the necessity of the hypotheses on the homomorphism φ.

This is joint work with Anders J. Frankild (University of Copenhagen) and Roger A. Wiegand (University of Nebraska-Lincoln). □

One topic that has become classical by now is the notion of finite Cohen-Macaulay type. The talk by Lars Christensen and Janet Striuli investigates a related concept and its relation to properties of singularities.

Finite Gorenstein representation type implies simple singularity

Lars Winther Christensen and Janet Striuli, University of Nebraska-Lincoln

Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Remarkable connections between the module theory of R and the character of its singularity emerged in the 1980s. They show how finiteness conditions on the category of maximal Cohen–Macaulay R-modules (the finitely generated modules whose depth equals the Krull dimension of R) characterize particular isolated singularities. We report on developments of these connections in several directions.

A local ring with only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules is said to be of finite Cohen–Macaulay (CM) representation type. By work of Auslander, every complete Cohen–Macaulay local ring of finite CM representation type is an isolated singularity.

Specialization to Gorenstein rings opens to a finer description of the singularities; it centers on the simple hypersurface singularities identified in Arnol′d's work on germs of holomorphic functions. By work of Buchweitz, Greuel, and Schreyer, Herzog, and Yoshino, a complete Gorenstein ring of finite CM representation type is a simple singularity.

In the talk we show how to avoid the *a priori* condition that R is Gorenstein by replacing finite CM representation type with a finiteness condition on the category $\mathcal{G}$ of modules of Gorenstein dimension 0. Over a Gorenstein ring, these modules are precisely the maximal Cohen–Macaulay modules, but they are known to exist over any ring, unlike maximal Cohen–Macaulay modules.

Our proof of this result employs a new notion of $\mathcal{G}$*-approximations*, which is close kin to the CM-approximations of Auslander and Buchweitz. Every module over a Gorenstein ring has a $\mathcal{G}$-approximation, and our proof goes via a strong converse: Assume there is a non-free module in $\mathcal{G}$; if the residue field k has a $\mathcal{G}$-approximation, then R is Gorenstein. □

Another talk on homological properties of rings was given by Hailong Dao, who presented new results on some classical homological questions.

On some homological questions over local rings

Hailong Dao, University of Utah

Consider the following classical results:
Theorem. Let (R, m, k) be a regular local ring and M, N be finite R-modules.

1. (Serre 1965) If $l(M \otimes_R N) < \infty$, then $\dim M + \dim N \leq \dim R$.
2. (Auslander 1961, Lichtenbaum 1966) For any integer $i \geq 0$, $\mathrm{Tor}_i^R(M, N) = 0$ implies $\mathrm{Tor}_j^R(M, N) = 0$ for all $j \geq i$.
3. (Auslander-Goldman 1960) Assume that M is a reflexive R-module. If $\mathrm{Hom}_R(M, M)$ is free then M is free.
4. (Auslander 1962) Assume that M is a reflexive R-module. If $\mathrm{Hom}_R(M, M) \cong M^{\oplus t}$ then M is free.
5. (Huneke-Wiegand 1997) Assume that M is a reflexive R-module. If $\mathrm{Hom}_R(M, M)$ satisfies (S_3) then M is free.

In this talk we will discuss some recent attempts to generalize these results to non-regular local rings. We will focus our attention on hypersurfaces and complete intersections, where the questions reveal some surprising connections. One particular result will be discussed:

Theorem. Let R be an admissible hypersurface (meaning $\hat{R}$ is a quotient of an unramified or equicharacteristic regular local ring by a nonzero element) with an isolated singularity. Assume that $\dim R > 2$ and is even. If M is a reflexive R-module such that $\mathrm{Hom}_R(M, M)$ satisfies (S_3), then M is free. □

One of the places where Algebraic Geometry and Commutative Algebra are most closely related is in the study of Castelnuovo-Mumford regularity for resolutions of graded modules. Marc Chardin discussed this topic for Tor modules.

The regularity of Tor over non-regular rings

Marc Chardin, Institut Mathematiques de Jussieu

In this lecture M. Chardin presented results about the behavior of Castelnuovo-Mumford regularity with respect to the functor Tor. One of the first motivations was to provide estimates on the regularity in a geometric context. This is for instance the content of the following result :

Theorem. *Let k be a field, $\mathcal{Z}_1, \ldots, \mathcal{Z}_s$ be closed subschemes of a closed subscheme $\mathcal{S} \subset \mathbf{P}_k^n$. Assume that $\mathcal{S}$ is irreducible with a singular locus of dimension at most 1. If $\mathcal{Z} := \mathcal{Z}_1 \cap \cdots \cap \mathcal{Z}_s \subset \mathcal{S}$ is a proper intersection of subschemes of $\mathcal{S}$ that are Cohen-Macaulay locally at points of $\mathcal{Z}$, then setting $r'_{\mathcal{S}} := \max\{\mathrm{reg}(\mathcal{S}) - 1, 0\}$, one has*

$$\mathrm{reg}(\mathcal{Z}) \leq \sum_{i=1}^{s} \max\{\mathrm{reg}(\mathcal{Z}_i), r'_{\mathcal{S}}\} + \lfloor (\dim \mathcal{S} - 1)/2 \rfloor r'_{\mathcal{S}}.$$

In particular, if $\mathrm{reg}(\mathcal{S}) \leq 1$, then $\mathrm{reg}(\mathcal{Z}) \leq \sum_{i=1}^{s} \mathrm{reg}(\mathcal{Z}_i)$.

Another motivation is the estimates obtained by Eisenbud, Huneke and Ulrich on the regularity of Tor modules over a polynomial ring, and their application to estimate the regularity of powers of an ideal. Their work was inspired by previous results on the regularity of products of ideals and of tensor products of modules by Conca and Herzog, Sidman and Caviglia. They have proved an upper bound for the regularity of $\mathrm{Tor}_i^R(M, N)$ in terms of $\mathrm{reg}(M)$ and $\mathrm{reg}(N)$, when R is a polynomial ring and $\mathrm{Tor}_1^R(M, N)$ is supported in dimension at most 1. M. Chardin presented the following extension :

Theorem. *Let S be a standard graded ring over a Noetherian local ring (S_0, m_0) and M, N be finitely generated graded S-modules. If M or N has finite projective dimension and $\mathrm{Tor}_i^S(M, N) \otimes_{S_0} S_0/m_0$ is supported in dimension at most one for $i \geq 1$, then*

$$\max_i\{\mathrm{reg}(\mathrm{Tor}_i^S(M, N)) - i\} = \mathrm{reg}(M) + \mathrm{reg}(N) - \mathrm{reg}(S).$$

An independent proof of this result in the case where $N = S_0$ is a field was given by Römer in the more general setting of positively graded algebras over a field. The hypothesis on the dimension of the support of all positive Tor modules (in place of only the first, in the polynomial case) is needed since Tor is not rigid when S is singular.

Some properties of multiple Tor modules are presented, in particular the rigidity of multiple Tor modules over a regular ring containing a field and a geometric condition for the vanishing of $\mathrm{Tor}_1^R(M_1, \ldots, M_s)$.

When I is a homogeneous ideal in a polynomial ring R over a field such that $\dim R/I$ is of dimension at most 1, it has been proved by Chandler and Geramita, Gimigliano and Pitteloud that $\mathrm{reg}(I^j) \leq j\mathrm{reg}(I)$ for any j. This has been refined by Eisenbud, Huneke and Ulrich who proved that in this situation $\mathrm{reg}(I^j) \leq \mathrm{reg}(I) + (j-1)(e-1)$ if I is generated in degrees at most $e-1$ and related in degrees at most e. A second refinement of the initial estimate, which holds for a homogeneous ideal I of a Noetherian standard graded ring S is presented :

Theorem. *Let I be an homogeneous S-ideal such that* $\dim(S/I) \otimes_{S_0} S_0/m_0 \leq 1$ *for any maximal ideal* $m_0 \in \mathrm{Spec}(S_0)$. *Set* $a_i(M) := \max\{j \mid H^i_{S_+}(M)_j \neq 0\}$. *Then, for any* $m \geq 0$,

$$\mathrm{reg}(S/I^{m+1}) \leq \max\{a_0(S/I) + b_0^S(I), (a_1(S/I) + 1) + \mathrm{reg}_1^S(I)\} + (m-1)b_0^S(I).$$

Notice that $a_1(S/I) + 1$ is the regularity of S/I^{sat}, where I^{sat} is the saturation of I with respect to the positive part of S. □

We include in this section two talks that were on the connections of Algebraic properties with continuous and analytic properties. The talk by Hal Schenk discusses algebraic properties of polyhedral complexes.

Splines on polyhedral complexes

Hal Schenck, Texas A&M University

In mathematics it is often useful to approximate a function f on a region by a “simpler” function. A natural way to do this is to divide the region into simplices, and then approximate f on each simplex by a polynomial function. A C^r-differentiable piecewise polynomial function on a d-dimensional simplicial complex $\Delta \subseteq \mathbb{R}^d$ is called a *spline*. Splines play a key role in geometric modeling, the finite element method for solving PDE’s, and in approximation theory.

It is possible to use polyhedra, rather than simplices, to subdivide a region. Splines on a polyhedral complex P have received relatively little attention (compared to the simplicial case), partly because the simplicial case fits very naturally into a homological framework. Billera and Rose observed that for any polyhedral complex, the splines occur as the kernel of a map between free modules; from this they obtain a bound on the Hilbert polynomial; other work on the polyhedral case has been done by Rose, Schumaker, and Yuzvinsky.

This talk describes an approach to the study of splines on a polyhedral complex which uses a certain specialized version of the dual graph of P; in particular, we show that the study of the first three coefficients of the spline module can be reduced to the study of certain subgraphs of the dual graph of P; these subgraphs arise from codimension two linear spaces which arise as intersections of the (linear hull) of the facets of P. This is joint work with Terry McDonald. □

Holger Brenner discussed finding continuous solutions to problems where classically one had looked for algebraic ones. An interesting point is that the solution can be given in terms of algebraic conditions.

Continuous solutions to algebraic forcing equations

Holger Brenner, University of Sheffield

We ask for a given system of polynomials $f_1, ..., f_n$ and f over the complex numbers $\mathbb{C}$ when there exist continuous functions $g_1, ..., g_n : \mathbb{C}^n \to \mathbb{C}$ such that $g_1 f_1 + ... + g_n f_n = f$. This condition defines the continuous closure of an ideal in a polynomial ring and more generally in any ring of finite type over $\mathbb{C}$. This closure sits inside the (weak sub) integral closure. We give inclusion criteria and exclusion results for this closure in terms of the algebraically defined axes closure. Conjecturally, continuous and the algebraically defined axes closure are the same, and we prove this in the monomial case by giving a combinatorial criterion which holds for both. □

5 Secant Varieties, Statistics, and Classical Projective Geometry

One of the most unexpected applications of Algebraic Geometry, in recent years, has been to the field of Statistics. There are many old and unsolved problems about the secant varieties of Segre varieties, and recently a new impetus for studying these problems has come from the realization that there would be exciting applications for solutions. In this conference we brought together mathematicians working on various aspects of these problems.

We present two talks that deal with the main topics of this section.

The talk by Seth Sullivant gave a striking example of the applications of Classical Geometry, and in particular of Secant Varieties, to Statistics.

Algebraic Geometry of Gaussian Bayesian Networks

Seth Sullivant, Harvard University

Given a directed acyclic graph G, the Bayesian network associated to G is the family of probability density functions that have a factorization of the form

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i | x_{\mathrm{pa}(i)})$$

where $\mathrm{pa}(i)$ is the set of parents of the vertex i in the directed acyclic graph G. In the case where we assume that the random vector $X = (X_1, X_2, \ldots, X_n)$ has a multivariate normal distribution (i. e., is a Gaussian random vector) the set of covariance matrices that can arise for this Bayesian network is a rational algebraic variety in the cone of positive definite covariance matrices. We provide an in-depth study of the vanishing ideal $I_G \subset \mathbf{C}[\sigma_{ij} : 1 \leq i \leq j \leq n]$ of this rationally parametrized set of covariance matrices. The parametrization of the set of covariance matrices turns out to be combinatorial in nature and is known in statistics as the trek rule.

Contained in the vanishing ideal of the model I_G, is the subideal CI_G which is generated by polynomial consequences of the conditional independence statements that any distribution in the Bayesian network associated to G must satisfy. The conditional independence ideal CI_G is generated by subdeterminants of the symmetric matrix Σ, that are determined by a combinatorial characterization called d-separation. A basic question is whether or not the conditional independence ideal CI_G is always equal to the vanishing ideal I_G. It turns out that there are already small graphs with only five vertices for which the inclusion $CI_G \subset I_G$ is strict.

Among the results discussed in this presentation is that for any tree T, it is always true that $I_T = CI_T$. The proof exploits the fact that for any tree I_T is a toric ideal and uses tools from the general theory of toric ideals.

The rest of the talk concerns the study of the models and graphs that arise when some of the random variables are hidden. One basic result shows that I_G has a 2-dimensional multigrading induced by a collection of upstream random variables. This, in turn, implies that generators of the ideal of the hidden variable model can be easily related to generators of the ideal I_G.

Finally, it is shown how classical varieties arise as special cases of hidden variable Gaussian Bayesian networks. In particular, it is shown how joins and secant varieties arise when the underlying directed acyclic graph has a partitioned structure in its hidden variables. Among the secant varieties that arise are secant varieties of toric degenerations of the Grassmannian of 2 planes $G_{2,n}$, and secant varieties of the squarefree Veronese variety. □

The topic of Secant Varieties was further discussed later that afternoon by Jessica Sidman, who dealt with them from a computational point of view.

Prolongations and computational algebra

Jessica Sidman, Mt. Holyoke University

In the late '90's Landsberg and Manivel drew intriguing connections between prolongation, a notion which first arose in the context of differential geometry, and the equations vanishing on secant varieties. Work of Sturmfels, Sullivant, et al in algebraic statistics, where secant varieties can be interpreted as statistical models, has renewed interest in secant varieties and their defining equations. Questions of Sturmfels provided the impetus for the work discussed in the talk.

The simplest form of the definition of prolongation follows: Let A be a vector space of homogeneous forms. The r-th prolongation of A, denoted $A^{(r)}$, is the space of all homogeneous forms of degree $d + r$ whose partial derivatives of order r are all contained in A.

The focus of the talk is to explain the connection between secant varieties and prolongation. In particular, we will see how the definition of prolongation can be reformulated algebraically in terms of polarization and how this version of the definition is related to forms vanishing on a secant variety.

This is joint work with Seth Sullivant.

A preprint can be found online: arXiv:math/0611696v2. □

In addition to the statistical applications, there has been a lot of work on the classical problems themselves. The group headed by Chiantini and Ciliberto (and mostly centered in Italy) has reexamined the work of the Italian 'masters' of the late 19th and early 20th century. They have recast the main classical results in modern terms and pushed them to unforseen levels. This Workshop was a perfect opportunity to have these ideas explained and discussed by experts.

The talk by Luca Chiantini considered the question of finding certain types of subvarieties in a general hypersurface and had, as its main ingredient, the study of secant and join varieties to the varieties of reducible forms.

Complete intersection subvarieties in hypersurfaces

Luca Chiantini, University of Siena

Which subvarieties Y does one find in a general hypersurface X of the complex projective space? The Noether-Lefschetz theorem describes the situation when codim$(Y, X) = 1 < \dim(X)$. For higher codimension, the problem is wide open.

In a joint research with E. Carlini and A. Geramita, we consider the problem of finding, in a general hypersurface X of degree d, a subscheme which is complete intersection of type $a_1, \ldots, a_s$ (the only relevant case being clearly $a_i < d$ for all i). Since complete intersections are arithmetically Gorenstein, when codim$(Y, X) = 2$ the problem is related with the theory of rank 2 vector bundles without intermediate cohomology on X, as well as to the pfaffian representation of general forms.

The problem has a nice interpretation in terms of secants and joins of some subvarieties of the variety of forms, namely the varieties of reducible forms. Using Terracini's lemma, the problem is then translated in a problem on the structure of certain Artinian algebras.

With this reduction, we are able to prove results on the subject. For example, we prove that for any choice of positive integers $a, b < d$, a general plane curve of degree d contains a complete intersection set of points of type a, b. The case of surfaces in the projective space turns out to be completely different. In general, one has no complete intersection of type a, b, c in a general surface of high degree. We are able to classify completely the (few) triples (a, b, c) such that a general surface of any degree $d > a, b, c$ contains a complete intersection set of points of type a, b, c. □

Another classical topic that has had impact in both Algebra and Geometry is that of varieties defined by determinantal ideals. Winfried Bruns described the variety defined by exterior powers.

The variety of exterior powers of linear maps

Winfried Bruns, University of Osnabrück

The lecture is based on joint work with Aldo Conca.

Let V and W be vector spaces of dimension m and n over a field K of characteristic 0. We investigate the Zariski closure X_t of the image Y_t of the map $\mathrm{Hom}_K(V,W) \to \mathrm{Hom}_K(\bigwedge^t V, \bigwedge^t W)$, $\phi \mapsto \bigwedge^t \phi$. In the case $t = \min(m,n)$, $Y_t = X_t$ is the cone over a Grassmannian, but for $1 < t < \min(m,n)$ one has $X_t \neq Y_t$. We analyze the $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$-orbits in X_t. It turns out that they are classified by two numerical invariants, one of which is the rank and the other a related invariant that we call *small rank*. Surprisingly, the orbits in $X_t \setminus Y_t$ arise from the images Y_u for $u < t$ and simple algebraic operations.

The classification of the orbits is based on explicit normal forms on the one hand, and a determination of the G-stable prime ideals in the coordinate ring A_t of X_t, the algebra generated by the t-minors of a generic $m \times n$-matrix in the polynomial ring $K[X]$ generated by the entries of the matrix. We investigate this algebra by means of its standard monomial basis.

In previous work with Conca we have shown that A_t is always normal and Cohen-Macaulay. For $t = 1$, one has the trivial case $A_t = K[X]$. The algebra A_t is also well-understood in the Grassmannian case $t = \min(m,n)$. If $t = m-1 = n-1$, then A_t is again isomorphic to a polynomial ring over K. Apart from these exceptional cases, in which A_t is a factorial domain, it has class group **Z** and is Gorenstein if and only if $1/t = 1/m + 1/n$. The singular locus of X_t is then formed by all elements of rank ≤ 1. □

The talk by Kuroda was on another very classical problem: the finite generation of algebras defined in certain ways. The original problem due to Hilbert asked whether certain subrings of polynomial rings, which included rings of invariants of algebraic groups, were finitely generated. The first counterexample was due to Nagata and was an application of Algebraic Geometry to this problem in Algebra. Since then there have been attempts to get simpler examples, of which the following talk is a culmination.

How to construct counterexamples to Hilbert's 14th problem easily

Shigeru Kuroda, Tokyo Metropolitan University

Let R be the polynomial ring in n variables over a field k for $n \in \mathbf{N}$, and K the field of fractions of R. Then, Hilbert's 14th problem asks whether the k-algebra $L \cap R$ is finitely generated whenever L is a subfield of K containing k. In 1950's, Zariski showed that the answer to this problem is affirmative if $\mathrm{trans.deg}_k(L) \leq 2$, while Nagata gave the first counterexample in case of $\mathrm{trans.deg}_k(L) = 4$ and $n = 32$. Here, $\mathrm{trans.deg}_k(L)$ denotes the transcendence degree of L over k. In 1990, Roberts found a different kind of counterexample. Following Roberts, we have made various kinds of new counterexamples. For example, we gave one having $\mathrm{trans.deg}_k(L) = 3$, one for which K/L is an algebraic extension, and a variety of derivations whose kernels are counterexamples to Hilbert's 14th problem.

In the talk, we give a simple method of converting a graded k-subalgebra of R with some conditions into a counterexample to Hilbert's 14th problem. As an application, we demonstrate how to construct (i) a counterexample with $[K : L] = d$ for each $d \geq 3$ when $n = 3$; (ii) a counterexample which is realized as the invariant field for an action of $\mathbf{Z}/2\mathbf{Z}$ on K for $n = 4$; (iii) a counterexample which is realized as the kernel of a derivation of K for $n = 4$. There commonly exist graded k-subalgebras which satisfy our conditions, so that we can get a large number of counterexamples by this method. □

Another active topic in this area is the existence of curves with given multiplicities through given points. Rick Miranda talked about his recent advance in that field. This study is strongly related to open problems involving secant varieties of the classically studied Segre and Segre-Veronese varieties.

Curves of degree 174 with ten points of multiplicity 55

Rick Miranda, Department of Mathematics, Colorado State University

Fix general points $p_1, \dots, p_n$ in the plane, and multiplicities $m_1, \dots, m_n$. Let

$$\mathcal{L} = \mathcal{L}_d(m_1, \dots, m_n)$$

be the linear system of plane curves of degree d having multiplicity at least m_i at p_i for each i. The *virtual dimension* of $\mathcal{L}$ is $v(\mathcal{L}) = d(d+3)/2 - \sum_i m_i(m_i+1)/2$ and the *expected dimension* is $e(\mathcal{L}) = \max\{-1, v\}$.

It is easy to see that if there exists a (-1)-curve C (on the blowup of the plane) such that $\mathcal{L} \cdot C \leq -2$ and $\dim(\mathcal{L}) \geq 0$, then $\mathcal{L}$ does not have the expected dimension.

Gimigliano-Harbourne-Hirschowitz Conjecture: This is if and only if: If no such (-1)-curve exists, then $\mathcal{L}$ has the expected dimension.

Gimigliano-Harbourne-Hirschowitz is true for $n \leq 9$. (Castelnuovo, 1891; Nagata, 1960; Gimigliano, Harbourne, 1986)

The virtual dimension of $\mathcal{L}_d(m^{10})$ is equal to -1 (the most delicate case) for (d, m) in the following table:

d	m	empty
3	1	easy: cubic through ten general points
19	6	posed by Dixmier, solved by Hirschowitz early 80s
38	12	Gimigliano's thesis
174	55	?
778	246	?
1499	474	?
6663	2107	?
$\vdots$	$\vdots$	?

For these linear systems, one expects there to be no such curves ($H^0 = 0$) and because $v = -1$, this is equivalent to having $H^1 = 0$ (for the line bundle on the ten-fold blowup of $\mathbb{P}^2$).

In this talk, the author explained the proof of the following:

Theorem: $\mathcal{L}_{174}(55^{10})$ is empty.

Theorem: $\mathcal{L}_d(m^{10})$ has the expected dimension if $d \geq (174/55)m$.

The proof is by an explicit degeneration of the plane to a configuration of nine surfaces. The degeneration is constructed by a sequence of blowups and blowdowns related to (-1)-curves on the components in the central fiber. $\square$

Finally, we present two talks on Hilbert schemes. First, Greg Smith described some geometric properties of multi-graded Hilbert schemes, a generalization of the classical singly graded Hilbert schemes.

Multigraded Hilbert schemes

Greg Smith, Queens University

There is a parameter space for all ideals in the polynomial ring $S := \mathbb{C}[x_1, \dots, x_n]$ with a fixed Hilbert function. What are the geometric properties of these spaces?

To be more precise, fix an abelian group A. An *A-grading* of S is induced by a group homomorphism $\deg\colon \mathbb{Z}^n \to A$. This map provides a decomposition $S = \bigoplus_{a \in A} S_a$ where S_a is the span of all the monomials of degree a in S. A homogeneous S-ideal I is *admissible* if $\dim_{\mathbb{C}}(S/I)_a < \infty$ for all $a \in A$; its *Hilbert function* $h_{S/I}\colon A \to \mathbb{N}$ is $h_{S/I}(a) := \dim_{\mathbb{C}}(S/I)_a$. M. Haiman and B. Sturmfels construct a quasiprojective scheme Hilb^h parametrizing all admissible S-ideals with Hilbert function $h\colon A \to \mathbb{N}$.

In general, the geometry of a multigraded Hilbert scheme Hilb^h is complicated. For example, R. Vakil shows that every singularity type appears in certain Hilb^h and F. Santos shows that there exists disconnected Hilb^h. In contrast, J. Forgarty proves that Hilb^h is smooth and irreducible when $n = 2$ and $A = 0$. Similarly, L. Evain proves that Hilb^h is smooth and irreducible when $n = 2$, $A = \mathbb{Z}$, both $\deg(x_1)$ and $\deg(x_2)$ are positive integers and h has finite support. Building on these results, M. Haiman and B. Sturmfels conjecture that Hilb^h is smooth and irreducible when $n = 2$. By extending L. Evain's methods, D. Maclagan and G. Smith prove this conjecture. □

The talk by Irena Peeva was also on the topic of Hilbert schemes and their relation to homological properties of rings.

Hilbert schemes and maximal Betti numbers

Irena Peeva, Cornell University

Throughout, S stands for the polynomial ring $k[x_1, \ldots, x_n]$ over a field k of characteristic 0. The ring S is graded by $\deg(x_i) = 1$ for each i. If J is a graded ideal, then the Hilbert function $h : \mathbf{N} \longrightarrow \mathbf{N}$ defined by $i \mapsto \dim_k J_i$ is an important numerical invariant. Lex ideals are special monomial ideals, defined in a simple combinatorial way. They play an important role in the study of Hilbert functions and syzygies.

Theorem 1.1. (over the polynomial ring S)

1. (1) (Macaulay) For every graded ideal J in S there exists a lex ideal L_J with the same Hilbert function.
2. (2) (Hartshorne) The Hilbert scheme $\mathcal{H}^h_S$, that parametrizes all graded ideals in S with a fixed Hilbert function h, is connected. More precisely, every graded ideal in S with Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h.
3. (3) (Bigatti, Hulett, and Pardue) Every lex ideal in S attains maximal Betti numbers among all graded ideals with the same Hilbert function.

Analogues of these results are proved over an exterior algebra by Kruscal-Katona, Peeva-Stilman, Aramova-Herzog-Hibi, and Mermin-Peeva-Stilman. Gasharov and Peeva prove the following analogues over a class of projective toric rings, which has received a lot of interest in Commutative Algebra and Algebraic Geometry: Veronese rings.

Theorem 1.2. Let $R = S/I$ be a Veronese toric ring.

1. (1) For every graded ideal J in R there exists a lex ideal L_J with the same Hilbert function.
2. (2) The Hilbert scheme $\mathcal{H}^h_R$, that parametrizes all graded ideals in R with a fixed Hilbert function h, is connected. More precisely, every graded ideal in R with Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h.
3. (3) Every lex ideal in R attains maximal Betti numbers among all graded ideals with the same Hilbert function.
4. (4) Every lex-plus-I ideal in S attains maximal Betti numbers among all graded ideals containing I with the same Hilbert function. □

The abstract by Adam Van Tuyl is in the area of classical geometry but discusses a purely algebraic idea, the property of being Arithmetically Cohen-Macaulay.

ACM sets of points in multiprojective space

Adam Van Tuyl,Lakehead University

Let $R = k[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{r,0}, \ldots, x_{r,n_r}]$ with $\deg x_{i,j} = e_i$ denote the $\mathbb{N}^r$-graded coordinate ring associated to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. If $\mathbb{X}$ is a finite set of reduced points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, and if $I_{\mathbb{X}}$ is the multi-homogeneous ideal of forms vanishing at $\mathbb{X}$, then it can be shown that the coordinate ring of $\mathbb{X}$, that is $R/I_{\mathbb{X}}$, has the property that $\dim R/I_{\mathbb{X}} = r$, but $1 \leq \operatorname{depth} R/I_{\mathbb{X}} \leq r$. A set of points $\mathbb{X}$ is called arithmetically Cohen-Macaulay (ACM) when $R/I_{\mathbb{X}}$ is Cohen-Macaulay, or in other words, when $\operatorname{depth} R/I_{\mathbb{X}} = r$. When

$r = 1$, then a set of points $\mathbb{X}$ in $\mathbb{P}^n$ is always ACM. However, when $r \geq 2$, it is possible that a set of points may fail to be ACM, so one is naturally lead to ask whether ACM sets of points can be classified.

For sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, three such classifications exist. The first classification, due to S. Giuffrida, R. Maggioni, and A. Ragusa (1992), classified ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ via their Hilbert functions. A second classification, based upon the geometry of the points, was later developed independently by the author (2003) and E. Guardo (2001). More recently, L. Marino (in progress) has shown how to use the notion of a multihomogeneous separator of a point (a multihomogeneous form F that pass through all but one of the points of $\mathbb{X}$) to classify ACM points in $\mathbb{P}^1 \times \mathbb{P}^1$.

In this talk, I will begin by recalling each classification. I will then show that the natural extension of each classification in $\mathbb{P}^1 \times \mathbb{P}^1$ to a general multiprojective space no longer holds. However, some new necessary and sufficient conditions for a set of points to be ACM will be presented. This talk is based upon joint work with Elena Guardo (University of Catania). □

6 Outcome of the Meeting

As we had intended, the meeting brought together researchers in many areas that connected Commutative Algebra with Algebraic Geometry, with the hope of promoting interaction between researchers in various different but related fields. The results were even better than we had hoped. There is a tremendous amount of research being carried out in these areas, and the participants profited greatly from hearing about and discussing the interactions between them. We expect this to lead to new developments and further expansion of the field.