1. **Minimal Surfaces and General Relativity**

The portions of Mathematical Relativity which were covered by the workshop include the study of the constraint equations, especially the study of black hole solutions. In the initial value formulation of the Einstein equations, the initial data are specified on a three dimensional manifold, and the data consists of a Riemannian metric (initial position) and a symmetric (0,2) tensor (which will be the second fundamental form of the spacelike hypersurface in the evolved spacetime) which plays the role of the initial velocity of the gravitational field. The Einstein equations impose an underdetermined set of nonlinear partial differential equations on the initial data called the constraint equations. This set of equations is entirely geometric in nature and arises from the Gauss and Codazzi equations for a spacelike hypersurface in a spacetime satisfying the Einstein equations. The content of the initial value problem is that any initial data set which satisfies the constraint equations evolves under the hyperbolic equations of motion to a local solution of the Einstein equations. Of course there are many unsolved questions concerning the long time behavior, but these do not fall under the topic of the workshop.

The minimal surface and mean curvature theory enters most directly in the Riemannian geometry of the constraint equations. The notion of trapped surface is defined in terms of mean curvature. When one has a trapped surface in the initial data it follows from theorems of Penrose and Hawking that the resulting spacetime will be singular, so such data is referred to as black hole initial data. It turns out that in an asymptotically flat data set which contains a trapped surface, there is always an outermost trapped surface which is often called an apparent horizon (or marginally outer trapped surface). Such a surface is a stable minimal hypersurface in the special case of initial data with zero initial velocity (often called the Riemannian case). The lecture of Galloway considered the question of the possible topologies which can occur for apparent horizons. He described a proof of the theorem (see [7], [8]) that in any dimension an apparent horizon is Yamabe positive in the sense that the induced metric can be conformally deformed to a metric of positive scalar curvature. This theorem generalizes a theorem of Hawking to the higher dimensional case. In three dimensions Hawking's theorem is a key step in the proof of the classical black hole uniqueness theorems. Galloway also described the recent work of Andersson/Metzger [1] and Eichmair [6] which solve the existence problem for the apparent horizon equation. An important question in this connection is whether the singularities which are known to arise in volume minimizing hypersurfaces of dimension 7 or more can arise generically in Einstein initial data.

Another important theorem which has been partially proven over the past decade is the Penrose inequality. This is a remarkable extension of the Positive Mass Theorem which provides a sharp lower bound for the mass when black holes are present. More precisely, this lower bound is given in terms of the area of an apparent horizon, and equality is achieved only for Schwarzschild metrics. The Riemannian Penrose inequality was first proven in three dimensions in 1997 by G. Huisken and T. Ilmanen [9] for the case of a single black hole. In 1999, H. Bray [2] extended this result to the general case of multiple black holes using a different technique. In his talk Dan Lee described his recent extension (joint with H. Bray [3]) of the Riemannian Penrose inequality to higher dimensions; precisely the inequality has been extended up to dimension less than 8. An interesting feature of the proofs of the Penrose inequality is that both proofs use weak solutions of evolution equations which are closely tied to the theory of volume minimizing hypersurfaces. The Huisken/Ilmanen proof which works only in the three dimensional case for a connected apparent horizon uses the $1/H$-flow while the Bray proof (which Bray and Lee have extended to higher dimensions) works for apparent horizons with multiple connected components, and uses a novel conformal flow of metric. The major outstanding problem in the area is the general Penrose inequality; that is, the corresponding inequality for arbitrary asymptotically flat initial data sets. The Riemannian Penrose inequality in dimensions 8 or more is also open and the difficulty is related to the possibility of singularities in volume minimizing hypersurfaces which may occur in this case.

The talk by Corvino described his work (see [4], [5]) on constructions of new solutions of the constraint equations which can be obtained by localized gluing methods; that is, gluing methods which produce smooth solutions which agree with the original solutions outside the gluing region. Such constructions are possible for the constraint equations because of its particular underdetermined structure. Corvino showed that any solution of the vacuum constraint equations with appropriate asymptotics can be deformed outside an arbitrarily large ball to a new solution which is identical with a Kerr (Schwarzschild in the Riemannian case) solution. Corvino also described his recent proof that the center of mass defined by Huisken and Yau [10] agrees with that defined by Corvino [4]. The Huisken/Yau definition is associated with a special foliation of the exterior region of the initial data by constant mean curvature spheres.

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2. Partial Differential Equations

Micah Warren from University of Washington spoke on "A priori estimates for special Lagrangian equations".

Abstract. We discuss recent a priori interior Hessian estimates for solutions of the special Lagrangian equation, when the equation has phase at least a certain value, or when the solution is convex. These equations include the sigma-2 equation in dimension three. The gradient graph of any solution is a minimizing Lagrangian surface. While Heinz showed in the 1950's that similar estimates hold for the sigma-2 (Monge-Ampere) equation in dimension two, Pogorelov showed that such estimates cannot hold for the sigma-3 (Monge-Ampere) equation in dimension three. This is joint work with Yu Yuan, partly also with Jingyi Chen.

The fully nonlinear special Lagrangian equation

$$F\left(D^2u\right) = \sum_{i=1}^{n} \arctan \lambda_i = \Theta \tag{2.1}$$

where λ_i are the eigenvalues of the Hessian D^2u, arises from the special Lagrangian geometry of Lawson and Harvey. The gradient graph $(x, Du\,(x))$ of the potential u is a Lagrangian submanifold in $\mathbb{R}^n \times \mathbb{R}^n$. The Lagrangian graph is called special when the phase, which is the argument of the complex number $\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_n\right)$, is constant Θ; that is, u satisfies equation (2.1). A special Lagrangian graph is a volume minimizing minimal submanifold in $\mathbb{R}^{2n}$.

In the 1950's, Heinz derived a Hessian bound for the two dimensional Monge-Ampère type equation, including (2.1) with $n = 2$. In the 1970's Pogorelov constructed irregular solutions to $\sigma_3\left(D^2u\right)\det(D^2u) = 1$ in dimension three, which were generalized to σ_k equations with $k \geq 3$ by Urbas. Hessian estimates for solutions with certain strict convexity constraints to Monge-Ampère equations and σ_k equation with $k \geq 2$ were obtained by Pogorelov and Chou-Wang respectively. Pointwise Hessian estimates in terms of certain integrals of the Hessian for σ_k equations and for special Lagrangian equation (1.1) with $n = 3, \Theta = \pi$ were produced by Urbas and by Bao-Chen, respectively. Recently, for (2.1) Hessian estimates have been obtained:

for convex solutions with a certain smallness assumption on the height in [4]; for a sharper bound when $n = 2$ in [5]; when $n = 3$ and $|\Theta| \geq \pi/2$, including the equation $\sigma_2(D^2u) = 1$ in dimension three, in [6], [7]. More recently, Hessian estimates for general convex solutions have been obtained in [1].

Open problems:

a) Whether one has Hessian control over the solutions to the special Lagrangian equation (2.1) with general phases in dimension three and higher, including $\triangle u = \det D^2u$ corresponding to $\Theta = 0$ and $n = 3$?

b) Derive a Hessian bound for the solutions to the quadratic Hessian equation $\sigma_2(D^2u) = \lambda_1\lambda_2 + \cdots = 1$ in dimension four and higher.

Joel Spruck from Johns Hopkins University spoke on "A half-space theorem for complete embedded cmc 1/2 surfaces in $\mathbb{H}^2 \times \mathbb{R}$ "

Abstract. The famous half-space theorem of Hoffman-Meeks says that a properly immersed minimal surface in $\mathbb{R}^3$ that is contained in a half-space must be a plane. We improve (in joint work with L. Hauswirth and H. Rosenberg) an analogous result for a complete properly embedded cmc 1/2 surface in $\mathbb{H}^2 \times \mathbb{R}$ (possibly with compact boundary).

Theorem 1. Let Σ be a complete properly embedded constant mean curvature $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$. Suppose Σ is asymptotic to a horocylinder C, and on one side of C. If the mean curvature vector of Σ has the same direction as that of C at points of Σ converging to C, then Σ is equal to C (or a subset of C if $\partial\Sigma \neq \emptyset$).

A strong motivation for the half space theorem is that it is used to prove the following result.

Theorem 2. Let Σ be a complete immersed constant mean curvature $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$. If Σ is transverse to the vertical Killing field $Z = \frac{\partial}{\partial t}$. Then Σ is an entire vertical graph over $\mathbb{H}^2$.

Such entire vertical graphs are plentiful. In fact using Theorem 2 and a construction of Fernandez-Mira, we have

Theorem 3. For each quadratic holomorphic differential on $\mathbb{C}$ or the unit disk, one associates an entire $H = 1/2$ graph.

The proof of Theorem 1 is based on the study of "horizontal graphs" over horocylinders, which satisfy the strange looking equation

$$(g^2 + g_t^2)g_{xx} - 2g_xg_tg_{xt} + (1 + g_x^2)g_{tt} = -g(1 + g_x^2) + \frac{W^3}{g^2} \tag{*}$$

Theorem 3 follows from the existence of catenoid-like solutions as in the proof of the Hoffman-Meeks half-space theorem.

Theorem 4. Let U be the annulus $U = B_{R_2} \backslash B_{R_1}$ with $R_2 \geq 2R_1$. Then for $\epsilon > 0$ sufficiently small (depending only on R_1), there exist constant mean curvature $H = 1/2$ horizontal graphs g^+ and g^- satisfying (*) on U with Dirichlet boundary data $g^\pm = 1 \pm \epsilon$ on ∂B_{R_1}, $g^\pm = 1$ on ∂B_{R_2}. Moreover $g^\pm$ is unique and varies continuously with the parameters ϵ, R_1, R_2 and $g^{\pm}$ tends to $1 \pm \epsilon$ uniformly on compact subsets as R_2 tends to ∞.

Bo Guan from Ohio State University spoke on "Complete conformal metrics with negative Ricci curvature on manifolds with boundary"

Abstract. Let $(\bar{M}^n, g)$, $n \geq 3$, be a compact smooth Riemannian manifold of dimension n with smooth boundary ∂M, $M = \bar{M} \setminus \partial M$ be the interior of $\bar{M}$, and let Ric_g denote the Ricci tensor of g. We are interested in the question of whether there exists a complete metric $\tilde{g}$ on M with negative Ricci tensor in the conformal class of g satisfying the equation:

$$\det(-Ric_{\tilde{g}}) = 1. \tag{2.2}$$

More generally, let f be a smooth symmetric function defined in a open convex symmetric cone $\Lambda \subset \mathbb{R}^n$ which contains $\Lambda_n^+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$ satisfying certain ellipticity structure conditions. Let $\Lambda^-[g]$ denote the collection of metrics $\tilde{g}$ on M in the conformal class of g such that $\lambda(Ric_{\tilde{g}}) = (\lambda_1, \cdots, \lambda_n)$, the eigenvalues of $\tilde{g}^{-1}Ric_{\tilde{g}}$, belongs to $-\Lambda$ everywhere on M.

Problem1. Find a complete metric $\tilde{g} \in \Lambda^-[g]$ on M with

$$f(-\lambda(Ric_{\tilde{g}})) = 1 \quad \text{in} \quad M. \tag{2.3}$$

In this talk we present some recent results from joint work with Huaiyu Jian and discuss open questions. Our result implies, in particular, that on any smooth domain in $\mathbb{R}$ contained in a half-space there exists a complete conformally flat metric with negative Ricci tensor satisfying equation (2.2).

Jaigyoung Choe from Korea Institute for Advanced Study spoke on "Capillary surfaces in a convex cone".

Abstract. Some capillary surfaces are known to be rigid, i.e., part of a sphere. Here are two known examples: a disk type capillary surface in a ball, and a disk type capillary surface with at most three edges in a domain bounded by planes and spheres. We have found more examples as follows. Let C be a convex cone in $\mathbb{R}^n$ a hypersurface in C which has constant higher-order mean curvature and is perpendicular to the boundary of C. Then S is a spherical cap. We will prove this using the Reilly formula.

Moreover, let C be a polyhedral cone in $\mathbb{R}^3$ and S a capillary surface in C with constant contact angle(not necessarily 90 degrees) and with at most 5 edges. Then S is a spherical cap. This can be proved by using Poincare-Hopf's theorem and Bonnet's parallel surface.

Finally Nitsche's result about a minimal disk in a ball which is perpendicular to the boundary sphere motivates the following problem: Let h be a harmonic function on a ball whose boundary values are equal to their normal derivatives. Show that they must be linear functions. We will prove this using Rellich's identity.

One of Choe's posed problems on harmonic functions with equal boundary Dirichlet and Neumann data was solved during a discussion in the workshop.

Pengfei Guan from McGill University spoke on "Isoperimetric inequality of quermassintegrals for starshaped domains"

Abstract. I will describe a recent joint work with Junfang Li. We give a simple proof of the isoperimetric inequality for quermassintegrals of non-convex starshaped domains. The proof is based on work of Gerhardt and Urbas for an expanding geometric curvature flow of hypersurfaces of $\mathbb{R}^{n+1}$ and the observation of a certain monotonicity property of isoperimetric constants.

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3. Calibrated submanifolds

Spiro Karigiannis spoke on "Moduli spaces of calibrated cycles in G_2 manifolds ".

G_2 manifolds [10] are a class of Ricci-flat manifolds with special holonomy, occurring in 7 real dimensions. They are analogous to Calabi-Yau 3-folds in many respects, and are of interest to physicists in M-theory and supergravity [1]. They admit two natural classes of calibrated submanifolds: the 3-dimensional associative submanifolds, and the 4-dimensional coassociative submanifolds. These are both absolutely volume-minimizing in their homology class. In joint work with Naichun Conan Leung [11], we prove that these submanifolds, together with unitary connections on them satisfying some special condition, are critical points of a naturally defined functional of Chern-Simons type. Specifically, the pair of an associative submanifolds together with a flat connection is such a critical point, as well as the pair of a coassociative submanifold together with a connection for which the trace of the curvature form is self-dual. Additionally, there is a special type of connection (called deformed Donaldson-Thomas connections) on the ambient 7-manifold which can also be interpreted as a critical point of the same functional.

An interesting and important open problem is to study the stability of these critical points. That is, are they non-degenerate, and if so, are they minima? This non-degeneracy is likely related to the smoothness of the moduli space of such objects. For example, it is known that the moduli space of coassociative submanifolds is smooth and unobstructed, whereas the situation is much worse for associative submanifolds [12]. They are in general obstructed and their infinitesimal deformations are the kernel of a twisted Dirac operator. The deformation theory of Donaldson-Thomas connections has not yet been analyzed.

Another important question that needs to be addressed is how the moduli of pairs of submanifolds and connections as described above relates to the analogous situation for Calabi-Yau 3-folds [3], [4]. If N is a Calabi-Yau 3-fold, and S^1 is a circle, then the product $M = N \times S^1$ is a G_2 manifold. Associative submanifolds include products of S^1 with a holomorphic curve in N, and coassociative submanifolds include products of S^1 with a special Lagrangian submanifold in N. The exact relationship between the two geometries is non-trivial, however, because the G_2 moduli involve a complicated mixing of the complex and Kähler moduli of the Calabi-Yau 3-fold.

Marianty Ionel talked about "Constructions of special Lagrangian submanifolds".

Abstract: Special Lagrangian submanifolds are a particular class of minimal submanifolds, introduced by Harvey and Lawson in the wider context of calibrated geometries. In this talk, I will describe the cohomogeneity one special Lagrangian 3-folds in both the deformed and the resolved conifolds. Our results give an explicit construction of the families of $SO(3)$ and T^2-invariant special Lagrangian submanifolds in these conifolds and describe their asymptotic behavior. The families of special Lagrangian submanifolds in the two conifolds approach asymptotically the same special Lagrangian cone. We will also describe a family of T^3-invariant coassociative 4-folds in the total space of the spin bundle of S^3, with the Bryant-Salamon G_2-metric.

Some open problems: 1. By moding out by appropriateS^1-actions on the spin bundle of S^3, one obtains the deformed and the resolved conifold. The relationship between the family of T^3- invariant coassociative 4-folds and the T^2-invariant special Lagrangian submanifolds constructed in the conifolds could be explored further using these actions. 2. Construct some symmetric explicit Cayley submanifolds in the spin bundle of S^4 endowed with the Bryant-Salamon $Spin(7)$-metric.

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4. Geometric flows

Curvature estimates are significantly important for the geometric evolution equations, such as the Ricci flow equation, etc. Gang Tian spoke on "Curvature estimates for Ricci flow in dimension 4". The following curvature estimates along the Ricci flow when the dimension of the manifold is 4 is stated and a proof is outlined.

Theorem. Suppose the Ricci flow equation

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

in $M \times [0,T)$, and $\dim M = 4$. Given $K > 0$, there exist $\epsilon, C = C(K)$ such that if

$$\int_{B_r(x,t_0)} |Rm|^2 \leq \epsilon$$

and

$$|Ric(g_0)| \leq K,$$

then

$$\sup_{B_{r/2}(x,t_0)} |Rm|_t \leq \frac{C}{t-t_0}$$

for $t \in (t_0, t_0 + r^2]$.

This theorem has a direct corollary as follows.

Corollary. Under the conditions

$$\int_{B_r(x)} |Rm|^2 dg \leq \epsilon$$

and $|Ric(g)| \leq K$, then $B_{r/4}(x)$ is diffeomorphic to U/Γ in $C^{1,\alpha}$ sense, where $U \subset \mathbb{R}^4$ is an open subset and $\Gamma \subset Iso(\mathbb{R}^4)$ is a finite group acting freely on $\mathbb{R}^4$.

This result can be viewed as a generalization of a theorem by Cheeger-Tian in [3], where the 4-manifolds are assumed to be Einstein.

G. Tian also proposed the following problem relevant to the the theorem:

Do all 4-manifolds with bounded Ricci curvature and finite Euler characteristic have finite topological type?

By the theorem for any sequence of 4-manifolds with bounded Ricci curvature and finite Euler characteristic, it will converge outside finitely many singularities. And if one could understand the topology around the singularities, then one could answer the above problems.

Yng-Ing Lee spoke on "Special solutions to Lagrangian mean curvature flow"

Abstract: In this talk, I will report on some special solutions to Lagrangian mean curvature flow constructed by me and my collaborators. The first category is eternal Brakke solutions. Recall that Brakke flow is a generalization of mean curvature flow, which is defined for varifolds, and an eternal solution is a solution defined for any t from negative infinity to infinity. Our solutions are (smooth) Lagrangian self-shrinkers for $t < 0$, Lagrangian cones for $t = 0$, and Lagrangian self-expanders for $t > 0$. Moreover, some of our solutions satisfy the additional property that every time slice is Hamiltonian stationary. When $n = 2$, our solutions resolve Schoen-Wolfson cones and can distinguish a $C_{2,1}$ cone from other cones. In higher dimension, we find Hamiltonian stationary cones which generalize Schoen-Wolfson cones, and eternal Hamiltonian stationary Brakke solutions which resolve these cones.

I will also talk about two other types of solutions. One type is translating solutions which include examples with arbitrarily small oscillation of the Lagrangian angle. These examples will play an important role in developing regularity theory in the Lagrangian mean curvature flow. Another type is other self-similar solutions. We have self-expanders with arbitrarily small oscillation of the Lagrangian angle and which are asymptotic to a pair of planes. Conversely, given any pair of Lagrangian planes with sum of characteristic angles less than $\pi/2$, we can always find such self-expanders asymptotic to this pair of planes. Examples of compact Lagrangian self-shrinkers are also obtained.

Jiayu Li from ICTP spoke on "Symplectic surfaces in K-E surfaces".

Abstract: In this talk I will review our recent results in symplectic translating solitons and symplectic critical surfaces in K-E surfaces.

In a Kähler surface M, one can define the Kähler angle α of an oriented surface Σ in M by $\omega|_\Sigma = \cos\alpha d\mu_\Sigma$, where ω is the Kähler 2-form of M and $d\mu_\Sigma$ is the area form of Σ in the induced metric. Σ is symplectic if $\cos\alpha \geq 0$ on Σ. Li-Han considered the critical points of the functional $\int_\Sigma 1/\cos\alpha d\mu_\Sigma$ in the space of symplectic surfaces. They derived the Euler-Lagrange equation for this functional and showed that it is an elliptic equation. Properties of this equation are then studied, for example, a formula on the number of the complex points which is similar to that in the minimal surface case . The gradient flow of this functional and evolution equations of various geometric quantities along this flow are also considered. In particular, if the initial surface is symplectic and closed, then it remains so along the flow as long as the flow exists.

For symplectic mean curvature flows, Li-Han show that $\sup|\alpha| > \frac{\pi}{4}\frac{|T|}{|T|+1}$, where α is the Kähler angle of a symplectic translating soliton with $\max|A| = 1$ and A is the second fundamental form and T is the direction in which the surface translates.

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5. The classical theory of minimal surfaces

New developments in the area of the classical theory of minimal surfaces were presented and discussed at the workshop - the topics included: new constructions of minimal submanifolds in Euclidean space and in spheres, classification of minimal surfaces, comparison between the second variation of area and energy for minimal surfaces, curvature estimates, etc... For example, Meeks presented recent work with Perez and Ros which settles the old question of classifying all properly embedded genus zero minimal surfaces in $\mathbb{R}^3$. Several of the mealtime group conversations revolved around discussions with Meeks on issues about and open questions on the classical theory of minimal surfaces [6]. Below is a highlight of the workshop talks.

Leobardo Rosales from UBC spoke on "Minimal immersions with prescribed boundaries". Recently L. Simon and N. Wickramasekera [8] introduced a PDE method for producing examples of stable branched minimal immersions in $\mathbb{R}^3$. This method produces q-valued functions u over the punctured unit disk in $\mathbb{R}^2$ so that either u cannot be extended continuously across the origin, or G the graph of u is a $C^{1,\alpha}$ stable branched immersed minimal surface. The present work gives a more complete description of these q-valued graphs G in case a discontinuity does occur, and as a result, we produce more examples of $C^{1,\alpha}$ stable branched immersed minimal surfaces, with a certain evenness symmetry.

Adrian Butscher of Stanford University spoke on "New constructions of submanifolds of the sphere which are critical points of the volume functional". If one searches for k-dimensional submanifolds with critical k-dimensional volume in a Riemannian manifold, then one is led towards elliptic partial differential equations involving the mean curvature vector of the submanifold. This talk presented new constructions ([1], [2]) of volume-critical submanifolds of the sphere in two contexts: hypersurfaces with constant mean curvature in spheres of any dimension; and Legendrian submanifolds in spheres of odd dimension that are stationary under variations preserving the contact structure. These are constructed by solving the associated elliptic PDE using singular perturbation theory. The analytic and geometric similarities between these two contexts was highlighted.

Mario Micallef of University of Warwick spoke on "Comparison between second variation of area and second variation of energy of a minimal surface". The conformal parameterisation of a minimal surface is harmonic. Therefore, a minimal surface is a critical point of both the energy functional and the area functional. This talk described joint work [3] with N. Ejiri which compares the Morse index of a minimal surface as a critical point of the area functional with its Morse index as a critical point of the energy functional. The difference between these indices is at most the real dimension of Teichmuller space. The methods for this comparison also allow Micallef and Ejiri to obtain surprisingly good upper bounds on the index of minimal surfaces of finite total curvature in Euclidean space of any dimension. They also bound the index of a minimal surface in an arbitrary Riemannian manifold by the area and genus of the surface, and the dimension and geometry of the ambient manifold.

Y.L. Xin of Fudan University spoke on "Curvature estimates for minimal submanifolds of higher codimension". Estimates of the Hessian of several smooth functions defined on Grassmannian manifold were derived. Based on these, curvature estimates for minimal submanifolds of higher codimension in Euclidean space were obtained, via the Gauss map ([9]). Thus, Schoen-Simon-Yau's results and Ecker-Huisken's result for minimal hypersurfaces are generalized to higher codimension. In this way, the results for Bernstein type theorems done by Hildebrandt-Jost-Widman and Jost-Xin could be improved.

William Meeks of the University of Massachusetts, Amherst, spoke on "The classification of embedded minimal planar domains in $\mathbb{R}^3$" (joint work with Joaquin Perez and Antonio Ros). Recently William Meeks, Joaquin Perez, and Antonio Ros [7] have succeeded in classifying all properly embedded genus 0 minimal surfaces in $\mathbb{R}^3$. Based on their previous results it remained to prove that the examples of infinite topology are the examples discovered by Riemann in 1860, called the Riemann minimal examples. The proof of the uniqueness of the Riemann minimal examples is in part related to the holomorphic integrability of the classical Shiffman function $S(M)$ which is a nonzero Jacobi function on a possible counterexample M. They relate the integrability to an evolution equation for the Gauss map of M which in turn can be related to the integrability of the KdV equation on the complex plane with initial Cauchy data. The proof of integrability depends on the theory of KdV hierarchy and algebro-geometric potentials. In the end they prove that $S(M)$ vanishes which means that M is foliated by circles and lines in a family of parallel planes, which by Riemann's earlier results implies M is a Riemann minimal example. Some related theoretical results were also discussed.

David Hoffman of Stanford University spoke on "Embedded Helicoidal minimal surfaces in $\mathbb{R}^3$ and $\mathbf{S}^2 \times \mathbb{R}$". In joint work [4] [5] with Brian White, they construct embedded genus-one helicoids in $\mathbb{R}^3$ by variational means without recourse to the Weierstrass representation or other function-theoretic methods. They are also able to show that important geometric properties of the examples they construct are shared by all other examples with sufficient symmetry. The talk described their construction of examples of embedded helicoidal minimal surfaces in $\mathbf{S}^2 \times \mathbb{R}$ of arbitrary genus.

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