Quadrature Domains and Laplacian Growth in Modern Physics

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Introduction. A great many physical processes involving moving boundaries can be reduced, after various idealizations, to the so-called Hele-Shaw problem. These are also known in the modern physics literature as Laplacian growth processes since the field equation governing is Laplace's equation and the subsequent interface motion is given by some surface derivatives of this field. Numerous physical phenomena that fall into this category: they include (but are not limited to) solidification processes [24], electrodeposition [10], viscous fingering [4], bacterial growth and the modelling of cancer cells [3]. The meeting in Banff in June 2007 focussed on the investigation of the dynamics and growth of unstable interfaces that appear as a result of such growth processes.

This field of research arguably originated in the 1940s [31, 12] and has attracted a great deal of attention recently in both the mathematical and physical communities due to newly found connections and applications to areas of classical physics and mathematical physics. These include integrable models, 2-dimensional quantum gravity and matrix models, the dynamics of quantum hall droplets, transport of 1-dimensional fermions, propagation of crystallization fronts and lightning propagation. The list of applications is clearly very wide-ranging.

The common mathematical background for these developments is the study of dynamics of the interface growth known as the Laplacian growth. The simplest model for this process is the displacement of the viscous fluid (say, oil) by a non-viscous one (referred to as *air* or, *water*, between two closely spaced horizontal plates (the Hele-Shaw cell). Due to the high viscosity of the oil, its incompressibility and the assumption that the flow takes place between two closely-spaced glass plates, the flow is governed by Darcy's law and the normal velocity of the interface is proportional to the normal gradient of the pressure. The zero viscosity of the air allows one to assume that the pressure is constant across the air domain.

If an interface develops a bump at some point in time the pressure gradient on that bump will be much larger than on the remaining "flatter" parts of the boundary of the interface. The bump therefore grows faster and will be quickly amplified during the growth process. This process is thus very unstable and can produce asymptotic shapes such as a "finger" [35], [5] or, within a discrete model, a fractal with characteristics similar to those obtained in Diffusion-Limited Aggregation processes, cf., e.g., [24, 10, 3]. The study of this growth phenomenon is currently enjoying a resurgence.

The mathematical model governing these and many other similar processes reduces to the following "simple" equation for the moving boundaries:

$$V(\xi) = \partial_n G_{D(t)}(\xi, a). \tag{1}$$

Here V is the normal component of the velocity of the moving boundaries $\partial D(t)$ of the time dependent domains $D(t) \subset \mathbf{R}^d$, $\xi \in \partial D(t)$, t is time, ∂_n is the normal component of the gradient, and $G_{D(t)}(\xi, a)$ is the Green function of the domain $D(t)$ for the Laplace operator with a unit source located at the point $a \in D(t)$.

Up to this point, the most rewarding theory from the applications perspective has been developed in two dimensions (2D). In two dimensions, the above equation can be rewritten as the area-preserving diffeomorphism identity

$$\Im\left(\bar{z}_t z_\phi\right) = 1, \tag{2}$$

where $z(t, \phi) := \partial D(t)$ is the moving boundary parameterized by $\phi \in [0,\, 2\pi]$ and conformal when analytically extended into the region $\Im\phi := \operatorname{Im}\phi \leq 0$ [12, 31]. The equation (2) possesses many remarkable properties among which the most noticeable one is the existence of an infinite set of conservation laws:

$$C_n(t) = \int_{D(t)} z^n \, dx \, dy = C_n(0), \tag{3}$$

where n runs over all non-negative [32] (non-positive [27]) integers in the case of a finite (infinite) domain $D(t)$, and an impressive list of exact time-dependent closed form solutions [37]. For a rather pleasing interpretation of conserved quantities C_n as coefficients of the multi-pole expansion of the fictitious Newtonian potential induced by the matter uniformly occupying the domain $D(t)$ see, e.g., [37].

It was established in [29] that the interface dynamics described by (2) is equivalent to the dispersionless integrable 2D Toda hierarchy [38], constrained by a so-called string equation. Remarkably, this hierarchy, being one of the richest existing integrable structures, describes an existing theory of 2D quantum gravity (see the comprehensive review [38] and references therein). The paper [29] generated a great deal of activity in apparently different mathematical and physical directions revealing profound connections between Laplacian growth and random matrices [20], the Whitham theory [21], and quadrature domains [30], [13, 11].

In short, Laplacian growth encapsulates a remarkable interconnection between mathematics, physics, and engineering. This means that any noticeable advance in one of these three branches of the subject often (if not always) produces subsequent discoveries in another. The BIRS meeting of 2007 gathered experts from all these fields, and this has reflected the highly interdisciplinary nature of the subject. The interaction among the participants was intense and the results, outlined below, are impressive.

The following publications reflect, if only partially, recent progress in the field obtained in a few fresh collaborations started at the BIRS 2007 workshop: [1, 2, 7, 14, 18, 25], see also the volume [6] and the survey [30].

1. Potential theory and Riemann surfaces. Recall that Laplacian growth is an interface dynamics where the boundary velocity equals the normal derivative of the Green function of the moving domain. Remarkably, this non-linear complex dynamics with infinitely many degrees of freedom possesses a complete set of conserved quantities (namely, the Richardson harmonic moments). Consequently, wide classes of *generalized quadrature domains* are preserved during the evolution. This implies in particular that so-called algebraic domains ("classical quadrature domains") remain algebraic.

A classical quadrature domain is an open subset Ω of the complex plane, which satisfies the quadrature identity

$$\int_\Omega f \mathrm{dArea} = u(f), \quad f \in L^1(\Omega, \mathrm{dArea}),$$

and u is a distribution with finite support (contained in Ω). By a doubling procedure such domains can be identified with a class of symmetric Riemann surfaces. Thus, Laplacian growth corresponds to certain kinds of dynamics of Riemann surfaces or, equivalently, of algebraic curves. A notable link between classical function theory on algebraic curves, elimination theory and quadrature domains was discovered in the article [14].

It is worth mentioning that an exact reconstruction algorithm of quadrature domains from finitely many power moments, or equivalent data, exists, see [11].

Much less understood is the "negative" Laplacian growth, under which the bounded domain either shrinks down to a potential theoretic skeleton of its original configuration, or breaks down due to singularity development on the interface. This process has a common fluid dynamics interpretation, namely a water/oil interface motion in a Hele-Shaw cell, where a fingering instability discovered by P. G. Saffman and G. I. Taylor in 1958, occurs [33]. Despite many efforts during subsequent years there are still unanswered questions in formulating a complete mathematical theory of it and quite a few contributions to the workshop were aimed in this direction.

2. Elliptic growth and the Beltrami operator. The occurence of the Laplace operator in mentioned above physical processes stems from the continuity and incompressibility conditions satisfied by a fluid involved in the potential flow. Specifically $\mathbf{v} = -\lambda \nabla p$, where $\mathbf{v}, \lambda$, and p are the fluid velocity vector field, conductivity, and scalar velocity potential, respectively:

$$\nabla \cdot \mathbf{v} = -\nabla \cdot \lambda \nabla p = -\lambda \nabla^2 p = 0.$$

As a first approximation, the conductivity λ was supposed to be constant, while generally it is not; therefore the major equation for growth has to be reconsidered as $\nabla \cdot \lambda(\mathbf{x}) \nabla p = 0$.

In [18] the authors present a natural extension of the Laplacian growth, where the Green function of $D(t)$ for the Laplace operator ∇^2 is replaced by the Green function of a linear elliptic operator,

$$L = \nabla \cdot (\lambda(\mathbf{x}) \nabla) - u(\mathbf{x}), \qquad \lambda(\mathbf{x}) > 0, \qquad \mathbf{x} \in \mathbf{R}^d. \tag{4}$$

Such a process, which was naturally cristened *an elliptic growth*, is clearly much more common in physics than the Laplacian growth. Consider, for instance, viscous fingering between viscous and inviscid fluids in the porous media governed by Darcy's law

$$\mathbf{v} = -\lambda \nabla p, \tag{5}$$

where λ is the filtration coefficient of the media and p is the pressure (equal to the Green function, $G_{D(t)}$). One can easily imagine a non-homogeneous media where the filtration coefficient λ is space-dependent. Such examples of elliptic growth, where the elliptic operator L has the form of the Laplace-Beltrami operator, $L = \nabla \cdot \lambda \nabla$, and λ is a prescribed function of $\mathbf{x}$, are called an elliptic growth *of the Beltrami type*. It is clear that all moving boundary problems other than viscous fingering with a non-homogeneous kinetic coefficient λ fall into this category.

From a mathematical point of view this process is the Laplacian growth occurring on curved surfaces instead of the Euclidean plane. In this case the Laplace equation is naturally replaced by the Laplace-Beltrami equation, and λ (that can also be a matrix instead of a scalar) is related to the metric tensor. There are several works addressing the Hele-Shaw problem on curved surfaces.

Another major source of examples of elliptic growth is related to screening effects, when $u \neq 0$, while λ is constant in (4). The simplest example of this kind is an electrodeposition, where the field p is the electrostatic potential of the electrolyte. It is known that in reality electrolyte ions are always locally surrounded by a cloud of oppositely charged ions. This screening modifies the Laplace equation for the electrostatic potential by adding to the Laplace operator the negative screening term, $-u(x)$, that stands for the inverse square of the radius of the Debye-Hukkel screening in the classical plasma. For the homogeneous screening u is a (positive) constant, so the operator L becomes the Helmholtz operator, while for the non-homogeneous case, when u is not a constant, L is a standard Schrödinger operator. Motivated by this example, the moving boundary problem for $L = \nabla^2 - u$ may be called an elliptic growth of *Schrödinger type*.

As was shown in [18] these rather general types of elliptic growth still retain remarkable mathematical properties, similar to those possessed by the Laplacian growth. A mixed case with a non-constant λ and non-zero u also shares similar properties but is less representative in physics and can always be reduced to one of the two former types of elliptic growth by a simple transformation.

It must be mentioned that in few prior works on elliptic growth an infinite number of conservation laws, regarded as extensions of (3), were identified as well, cf. [37, 28]. An integrable example in 2D, corresponding however to a very special choice of the conductivity function, $\lambda(\mathbf{x})$, was explicitly constructed in [25]. The elliptic growth there essentially reduces to the well-known Calogero-Moser integrable system.

3. Stochastic analysis and fractal growth. Very recently, an *integrable model* for the stochastic Laplacian growth with finite-size deposited particles within the framework of so-called *Loewner chains* was developed. As a consequence, it is expected to recover universal geometric characteristics, such as the multifractal spectrum of the growing clusters. Notably, this process retains integrability, despite its randomness.

In another important work combining stochasticity with integrability [16], the authors have obtained a list of surprising results connecting random entities and the tau-function - a powerful concept in the theory of integrable systems. Taken together with the above mentioned results in Laplacian growth, such nontrivial interconnections between integrability and randomness provide a constant source of new ideas.

4. Laplacian growth as a large N limit of random matrices spectra. Some deep links between the stochastic Laplacian growth and the theory of random matrices are discussed in the survey [30]. As it is often the case in other applications of random matrices, this connection sheds new light on older classical problems. To give a single example, an important observation was made in [19] and developed in [36] (see also review [39]) that the Laplacian growth can be simulated by the evolution of an averaged spectrum of normal random matrices as a function of a re-scaled size of matrices from the statistical ensemble, when the size of a matrix, N, goes to infinity.

The main observation is surprising: the evolution of the support of the eigenvalues can be treated as the Laplacian growth of the domain. Namely, it behaves exactly as an air bubble in the Hele-Shaw cell (with zero surface tension). Considering random matrices and the more general 'beta-ensembles' with the probability measure

$$\prod_{j<k}^{N} |z_j - z_k|^{2\beta} \prod_{l=1}^{N} d\mu(x_l, y_l),$$

(here $d\mu(x,y)$ is a smooth measure in the plane), a natural framework is calling to be developed, with the aim at solving the stochastic version of Laplacian growth. Clearly, the large N approximation is by no means enough for this purpose, so one should properly take into account $1/N$-corrections and understand the structure of the whole $1/N$-expansion. This direction is now under intense development, and it can be fair to state that it has started at the BIRS 2007 workshop.

5. Complex orthogonal polynomials. As mentioned above, the (renormalized) eigenvalues of ensembles of random normal matrices constrained by simple external field potentials are known, in the limit as the size of the matrix tends to infinity, to occupy regions that are generalized quadrature domains. In this way, the methods of statistical physics intersect with ideas from function theory and approximation theory with very surprising results (cf. [1, 30]). In particular, it has been proved that the geometry of the limiting domain, encoded in its Schwarz function, determines the cluster of zeros of some canonically associated complex orthogonal polynomials. The resulting potential theoretic skeleton of the limiting domain remains quite mysterious, and it is currently under intense investigation by a number of researchers.

6. Applications to classical physics. The same mathematics of Laplacian growth, involving conformal mapping theory, analytical/numerical uniformization and function theory on compact Riemann surfaces, also arises in a rich array of quite separate problems in classical physics: in fluid mechanics, for example, it arises in the study of free surface Stokes flows and in vortical solutions of the Euler equations. A review of many different physical problems, all arising just within the field of classical fluid dynamics, where quadrature domains arise has recently been compiled [9]. Most recently, Laplacian growth models have been found to be relevant to describing ionization processes in electrical streamers [26] – a physical problem where Maxwell's equations govern the physics. Such cross-disciplinary applications of the mathematics of Laplacian growth are many and varied and new instances are continually being uncovered.

7. Numerical simulations and industrial applications. It is well-known that, in the continuous Laplacian growth problem, the initial value problem can, under certain conditions, be ill-posed. Owing to this ill-posedness, when small regularization effects *are* included – for example, by including surface tension effects – any numerical method for resolving the subsequent dynamics encounters a variety of challenges and much research has gone into resolving these numerical issues over recent years. Other challenging mathematical problems arise in the asymptotic analysis of such problems. For example, a long-standing problem that was eventually solved in the 1980's involved the *selection mechanism* for the Saffman-Taylor viscous fingering problem. The solution to this problem gave birth to a new area of asymptotic analysis now known as *asymptotics beyond all orders* [34] owing to the role of exponentially small terms in picking out allowable solutions. Many challenges in both the numerical and asymptotic analysis of Laplacian growth problems remain and are the subject of ongoing work.

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