Global attraction to solitary waves in nonlinear dispersive Hamiltonian systems

Vladimir Buslaev (St. Petersburg University),
Andrew Comech (Texas A&M University),
Alexander Komech (University of Vienna),
Boris Vainberg (UNC - Charlotte)

May 20, 2008 – May 30, 2008

1 Overview of the Field

The long time asymptotics for nonlinear wave equations have been the subject of intensive research. The modern history of nonlinear wave equations starts in nonlinear meson theories [Sch51a, Sch51b]. The well-posedness was addressed in early sixties by Jörgens [Jör61] and Segal [Seg63a]. Later Segal [Seg63a, Seg63b], Strauss [Str68], and Morawetz and Strauss [MS72], where the nonlinear scattering and local attraction to zero were considered.

Global attraction (for large initial data) to zero may not hold if there are stationary or *quasistationary* solitary wave solutions of the form

$$\psi(x,t) = \phi(x)e^{-i\omega t}, \qquad \textit{with} \quad \omega \in \mathbb{R}, \quad \lim_{|x|\to\infty} \phi(x) = 0. \tag{1}$$

We will call such solutions *solitary waves*. Other appropriate names are *nonlinear eigenfunctions* and *quantum stationary states* (the solution (1) is not exactly stationary, but certain observable quantities, such as the charge and current densities, are time-independent indeed).

According to "Derrick's theorem" [Der64], time-independent soliton-like solutions to Hamiltonian systems, under rather general assumptions, are unstable. On the other hand, *quasistationary* solutions may be stable. This is caused by the additional conservation laws, which may prevent a slightly perturbed solitary wave from tumbling in the direction of lower energy states. This stimulated the study of the existence and stability properties of solitary waves in the Hamiltonian systems with symmetries.

Existence of solitary waves was addressed by Strauss in [Str77], and then the orbital stability of solitary waves in a general case was proved in [GSS87]. The asymptotic stability of solitary waves was considered by Soffer and Weinstein [SW90, SW92], Buslaev and Perelman [BP93, BP95], and then by others.

The existing results suggest that the set of orbitally stable solitary waves typically forms a *local attractor*, that is, attracts any finite energy solutions that were initially close to it. Moreover, a natural hypothesis is that the set of all solitary waves forms a *global attractor* of all finite energy solutions.

2 Current State of Things: Example

We are interested in the stability properties of solitary waves and long-time asymptotics of finite energy solutions. This field remains very active for the last thirty years, and yet many questions are not understood. Let us illustrate the state of things on the example of the Nonlinear Klein-Gordon Equation:

NLKG: $$\partial_t^2 u(x,t) = \partial_x^2 u + F(u), \qquad x \in \mathbb{R}, \quad t \geq 0, \quad u(x,t) \in \mathbf{C}.$$

This equation describes oscillations of a string with certain elasticity properties (represented by a smooth function $F(u)$, "the nonlinearity"). The string is infinite and stretched along the x-axis. Real and imaginary parts of $u(x,t)$ are the y- and z-coordinates of the piece of the string above the point x at the moment t. We assume that F is smooth and satisfies $F(e^{is}u) = e^{is}F(u)$, $s \in \mathbb{R}$, so that the equation is $\mathbf{U}(1)$-invariant. For a particular $F(u)$, one would like to know:

1. *Are there solutions of the form $\phi_\omega(x)e^{-i\omega t}$ with $\phi_\omega(x)$ localized ("solitary waves")?*
2. *Which solitary waves are orbitally stable (so that small perturbations do not grow)?*
3. *Which solitary waves are asymptotically stable (so that small perturbations disperse)?*
4. *As $t \to \infty$, does* any *finite energy solution look like outgoing solitary waves?*

These most natural questions in the context of nonlinear dispersive $\mathbf{U}(1)$-invariant Hamiltonian systems are the central questions of the PDE Theory. It is mainly due to the research of W. Strauss and his school (since the seventies) that **Questions 1, 2** are understood rather well. In particular, it is known that for the same nonlinearity some solitary waves could be stable while others could be unstable.

The asymptotic stability (**Question 3**) could be viewed as the *local attraction* to solitary waves. At present, not much is known about asymptotic stability, especially in the translation-invariant case; only the nonlinear Schrödinger equation has been studied in the works by Buslaev and Perelman; this was developed by Cuccagna and others. (No results for NLKG as of yet.) The asymptotic stability is proved under very strong assumptions on the spectrum and the order of vanishing of the nonlinearity. At the same time, one expects asymptotic stability for any orbitally stable solitary wave.

The last question (**Question 4**) is about the *global attraction*. Namely, we would like to know whether the system has a *finite-dimensional attracting set*, and whether it is formed by solitary waves. Properties of global attractors are well-understood for the *dissipative* systems (such as Navier-Stokes equation). Yet, essentially nothing is known even for NLKG in 1D.

Both the asymptotic stability and the global attraction are based on the "disersive damping": the nonlinearity increases the spectrum of the perturbation, creating the dispersive waves that carry the excess energy and charge away. This mechanism (called "friction by dispersion") is still not described rigorously.

3 Presentation Highlights

The core of the focussed research group was two intensive lecture courses by Professors Buslaev and Vainberg. Vladimir Buslaev presented a series of lectures on completely integrable systems, concentrating on the example of the completely integrable cubic Schrödinger equation. Such systems are the only example of dispersive models where the solitary asymptotics are known.

Boris Vainberg presented a series of lectures on quasiclassical asymptotics. Our expectation is that these methods could be applied to solitary asymptotics, giving a very detailed description of higher density – low energy electron waves in the magnetic field, when the nonlinear quantum effects become important and Vlasov-type classical particle description is no longer applicable.

Both these courses were exceptional in addressing very deep aspects of the theory and also giving the hands-on experience with the main technical methods.

Andrew Comech gave a minicourse on convergence to solitary waves in nonlinear nonlintegrable dispersive models based on the Klein-Gordon equation. The convergence to solitary waves in these models is the only existing result when the convergence of any finite energy solution to one of solitary waves is caused by dispersion.

Besides the lecture courses, Alexander Komech, Elena Kopylova, Galina Perelman, and David Stuart presented the most recent developments in the theory of local and global attraction to solitary waves.

4 Scientific Progress due to the Focused Research Group

The discussions during the workshop resulted in an active research in the following directions:

1. Global attraction to solitary waves. The results on global attraction have been proved for several systems:
   (a) For the 1D Klein-Gordon equation coupled to nonlinear $\mathbf{U}(1)$-invariant oscillators [KK07a, KK08b];
   (b) For the nD Klein-Gordon equation coupled to nonlinear $\mathbf{U}(1)$-invariant oscillator, with the mean field interaction [KK07b, KK08a].
2. Asymptotic stability of solitary waves. The convergence to solitary waves for the slightly perturbed initial data has been proved for the following systems:
   (a) For the wave equation coupled to a particle [IKV07];
   (b) For 1D Schrödinger equation coupled to nonlinear $\mathbf{U}(1)$-invariant oscillator [BKKS08] with no nonzero discrete spectrum;
   (c) For 1D Schrödinger equation coupled to nonlinear $\mathbf{U}(1)$-invariant oscillator [KKS08] with a nonzero discrete spectrum.
3. Dispersive decay in weighted energy norm. Such a decay has been proved for 2D discrete Schrödinger and Klein-Gordon equations [KKV08].
4. Discussions in the workshop encouraged David Stuart to start work on the asymptotic stability of Ginzburg-Landau vortices in the Chern-Simons-nonlinear Schrödinger system, using some ideas developed in the context of nonlinear Schrödinger equations. However, in the Chern-Simons context there is a richer dynamical structure including nontrivial vortex interactions, which can be described precisely in the self-dual limit.

The above results were reported during the Miniworkshop *"Global Attractors in Hyperbolic Hamiltonian Systems"* at the 5th European Congress of Mathematics in Amsterdam, July 2008.

Two more topics were actively discussed during the workshop, and are the subject of ongoing research:

1. Stability of solitary waves in nonlinear Dirac equation. It was suggested (and now numerically confirmed [CC08]) that the small amplitude solitary wave solutions to the nonlinear Dirac equation in 1D are (generically) spectrally stable.
2. Wave-particle dualism and the solitary asymptotics in the cathode rays. The participants found a way to advocate the possibility of explaining the electron diffraction results (Davisson – Germer experiment) in terms of solitary wave asymptotics for Maxwell-Dirac system.

References

[BKKS08] V. S. Buslaev, A. I. Komech, E. A. Kopylova, and D. Stuart, On asymptotic stability of solitary waves in Schrödinger equation coupled to nonlinear oscillator, *Comm. Partial Differential Equations* **33** (2008), 669–705.

[BP93] V. S. Buslaev and G. S. Perel′man, Scattering for the nonlinear Schrödinger equation: states that are close to a soliton, *St. Petersburg Math. J.* **4** (1993), 1111–1142.

[BP95] V. S. Buslaev and G. S. Perel′man, On the stability of solitary waves for nonlinear Schrödinger equations, in *Nonlinear evolution equations*, vol. 164 of *Amer. Math. Soc. Transl. Ser. 2*, 75–98, Amer. Math. Soc., Providence, RI, 1995.

[CC08] A. Comech and M. Chugunova, On stability of solitary waves in nonlinear Dirac equation on a line (2008), preprint.

[Der64] G. H. Derrick, Comments on nonlinear wave equations as models for elementary particles, *J. Mathematical Phys.* **5** (1964), 1252–1254.

[GSS87] M. Grillakis, J. Shatah, and W. Strauss, Stability theory of solitary waves in the presence of symmetry. I, *J. Funct. Anal.* **74** (1987), 160–197.

[IKV07] V. Imaykin, A. Komech, and B. Vainberg, On scattering of solitons for wave equation coupled to a particle, in *Probability and mathematical physics*, vol. 42 of *CRM Proc. Lecture Notes*, 249–271, Amer. Math. Soc., Providence, RI, 2007.

[Jör61] K. Jörgens, Das Anfangswertproblem im Grossen für eine Klasse nichtlinearer Wellengleichungen, *Math. Z.* **77** (1961), 295–308.

[KK07a] A. I. Komech and A. A. Komech, On global attraction to quantum stationary states II. Several nonlinear oscillators coupled to massive scalar field, *MPI MIS Leipzig preprint* **17/2007** (2007).

[KK07b] A. I. Komech and A. A. Komech, On global attraction to quantum stationary states III. Klein-Gordon equation with mean field interaction, *MPI MIS Leipzig preprint* **66/2007** (2007).

[KK08a] A. I. Komech and A. A. Komech, Global attraction to solitary waves for Klein-Gordon equation with mean field interaction, *Ann. Inst. H. Poincaré Anal. Non Linéaire* (2008), to appear.

[KK08b] A. I. Komech and A. A. Komech, Global attraction to solitary waves in models based on the Klein-Gordon equation, *SIGMA* **4** (2008), Proceedings of the Seventh International Conference "Symmetry in Nonlinear Mathematical Physics" (June 24-30, 2007; Institute of Mathematics, Kyiv, Ukraine).

[KKS08] A. I. Komech, E. A. Kopylova, and D. Stuart, On asymptotic stability of solitary waves for schrödinger equation coupled to nonlinear oscillator, II, *submitted to* SIAM J. Math. Analysis (2008), arXiv 0807.1878, submitted to *SIAM J. Math. Analysis*.

[KKV08] A. I. Komech, E. A. Kopylova, and B. R. Vainberg, On dispersive properties of discrete 2D Schrödinger and Klein-Gordon equations, *J. Funct. Anal.* **254** (2008), 2227–2254.

[MS72] C. S. Morawetz and W. A. Strauss, Decay and scattering of solutions of a nonlinear relativistic wave equation, *Comm. Pure Appl. Math.* **25** (1972), 1–31.

[Sch51a] L. I. Schiff, Nonlinear meson theory of nuclear forces. I. Neutral scalar mesons with point-contact repulsion, *Phys. Rev.* **84** (1951), 1–9.

[Sch51b] L. I. Schiff, Nonlinear meson theory of nuclear forces. II. Nonlinearity in the meson-nucleon coupling, *Phys. Rev.* **84** (1951), 10–11.

[Seg63a] I. E. Segal, The global Cauchy problem for a relativistic scalar field with power interaction, *Bull. Soc. Math. France* **91** (1963), 129–135.

[Seg63b] I. E. Segal, Non-linear semi-groups, *Ann. of Math. (2)* **78** (1963), 339–364.

[Str68] W. A. Strauss, Decay and asymptotics for $\Box u = f(u)$, *J. Functional Analysis* **2** (1968), 409–457.

[Str77] W. A. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.* **55** (1977), 149–162.

[SW90] A. Soffer and M. I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations, *Comm. Math. Phys.* **133** (1990), 119–146.

[SW92] A. Soffer and M. I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations. II. The case of anisotropic potentials and data, *J. Differential Equations* **98** (1992), 376–390.