REPORT ON THE BIRS WORKSHOP 06W5090 "GEOMETRIC AND NONLINEAR ANALYSIS"

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1. Objectives of the workshop

The objective of this workshop was to cover a wide range of topics in nonlinear partial differential equations, both on the theoretical and on the applied point of view. As the title suggests, we insisted mainly on pde's arising in geometric analysis.

The idea was to make meet together people working in various topics of this very wide subject. We insisted mainly on conformal geometry and fully nonlinear equations, without neglicting other topics. As a matter of fact, the main objective of this workshop was to discuss recent developments in many branches of geometric analysis and to stress on possible new applications.

However, some new techniques, geometric or not, developed by teams concerned with these applications, do not make their way to others who may be using similar techniques but on different types of problems. One particularity, which is common to these very active areas, is the appearance of singularities. These singularities are of various type, often specific to each domain, and always need to be understood. Therefore, the other objective of this workshop was to mix senior experts and young researchers from either a more geometric flavour or a more "pure" pde flavour. In particular concerning young researchers, we were very happy to see that PhD students and post-docs were highly interested in this event and attended the workshop with pleasure.

The organizers were quite delighted to see that these objective were fullfilled. Many researchers interacted with each others and discovered new branches and new methods. Such an interaction provided an opportunity to exchange ideas from various sensibilities so that new applications could be developed. The balance between talks and free time, in the inspiring and majestuous environment of the Banff National Park, allowed the researchers in all these fields to interact together and spread their techniques one to each other.

The organizers thank the BIRS staff for their particular efficiency to manage the station and for always being happy to oblige.

2. Thema treated

2.1. **Conformal Geometry.** Conformal geometry in the large was clearly the most considered theme during the workshop. We focused on various conformal invariants. The most natural invariant is the scalar curvature, which is deeply related to the topology of two-dimensional surfaces: it also enjoys conformal invariance in dimension $n \geq 3$, and the prescription of the scalar curvature in a conformal class has been the target of investigations for decades, generalizing the Yamabe problem.

There has been some recent progresses towards the understanding of the structure of the Yamabe equation. Indeed, the difficulty in this question is due to the conformal invariance of the equation: this invariance allows bubbling to occure and singularities to appear. The difficulty is to tackle these singularities. F.Coda Marques presented his recent work concerning compactness of the Yamabe equation, a work that is a step towards the fool analysis of the equation. O.Druet presented the asymptotics for a critical equation in dimension two which is a more general version of the Yamabe equation. The Yamabe equation is related to the study of nonlinear PDEs of second order with critical growth: therefore, the framework of the Yamabe equation is naturally applied to various nonlinear PDEs, geometric or not. In particular, despite the equation is not geometric, the bubbling appears in the MEMS equations studied by N.Ghoussoub: a control on the bubbling allows to prove multiplicity results and to use bifurcation methods.
The questions related to the conformal Yamabe equation can be generalized in two directions. If one considers the scalar curvature as attached to a second order problem, one will find interest in higher order conformal invariants, like the $Q-$curvature for fourth-order operators related to the conformal Paneitz-Branson operator. In this spirit, A.Malchiodi presented a recent result concerning the prescription of this curvature in dimension four in the compact setting, and H.-C.Grunau was interested in multiplicity of metrics in the noncompact setting of the hyperbolic space.
By the way, following Graham and al., it is possible to construct conformal invariants attached to arbitrary high order linear operators related to conformal infinity of noncompact manifolds. This theme was studied by C.Guillarmou who related the determinants of these operators to the Selberg's zeta function. Also in the context of the conformal infinity of hyperbolic spaces, R.Mazzeo presented existence and uniqueness of foliations with prescribed curvature. Finally, D.Raske considered compactness issues for an extension of the Yamabe equation at higher order.
The direction described above concerned linear operators. If one sees the scalar curvature as the sum of the eigenvalues of the Schouten tensor, one can consider other symmetric functions of the eigenvalues, which leads to fully nonlinear equations of second order, that is the σ_k equation. We had many talks in this directions. In particular, Z.-C.Han discussed the loss of compactness associated to this equation, non-existence results (in the spirit of Kazdan-Warner) and existence results for the prescription of σ_k. Y.Ge discussed the prescription of another invariant via the flow.

2.2. **"Non-conformal" Differential Geometry.** Despite conformal geometry was dominating, it was not the only aspect of geometry considered in this workshop. In her talk, S.-Y.A.Chang reported on regularity results concerning Bach-flat metrics on four manifolds with boundary: indeed, she provided an $\epsilon-$regularity theorem which is delicate for points on the boundary. S.Kim also considered Bach-flat four-manifolds, but when they are noncompact and complete: he presented a rigidity result by proving that when the curvature is small in the L^2-sense, then these manifolds are flat. Still concerning the structure (and also conformal geometry), J.Qing established a classification of degenerate Riemannian metrics. P.Yang discussed regularity issues for conformal minimal surfaces in pseudo-Hermitian geometry.
This concerned Riemannian geometry. We also had contribution in Spin geometry by E.Humbert. F.Pacard reported on existence of extremal Kähler metrics on some specifice manifolds, and concerning contact geometry, RCAM Van der Vorst

presented the proof of the Weinstein's conjecture for closed characteristic in the noncompact setting.
At the intersection of hyperbolic equations and geometry, M.Dafermos presented in his talk some new decay estimates in the context of black hole exterior spacetime: an important question which is related to the stability of black hoels in general relativity.

2.3. **Fully nonlinear equations.** The two main problems discussed in the conference were the fully-nonlinear version of the Yamabe problem (the σ_k equation) which connects this theme with the preceding conformal geometry theme, and the optimal transportation theory. The recent developments in transport, and especially its new applications, like in functional inequalities or evolutions pde's, are naturally of perticular interest. This very active area had a natural place in this conference. We refer to the preceding contributions by Z.-C.Han and Y.Ge for issues related to the σ_k equation in conformal geometry. Concerning optimal transportation, we had a general survey by R.McCann: this survey explained really nicely the history, difficulties and issues related to this very old subject having its roots in the Monge-Kantorovich problem. Regularity issues being quite intricate for fully-nonlinear equations, R.McCann also detailed optimal results, examples and counterexamples.

2.4. **Other subjects...** Some of the contributions were very general. Indeed, the techniques and ideas developed or them are really an important potential for other areas in Mathematics.
Y.Brenier presented a very elegant L^2 approach to hyperbolic consevration laws. A remarkable point in this formulation is that it permits to recover directly many existing results and that it can also be applied to many other problems.
M.Struwe revisited the regularity theory for harmonic maps. Indeed, in a joint work with T.Rivière, he proved the well-known regularity results with a completely new point of view and new techniques via Gauge theory. One advantage here is that this very elegant point of view applies to an important class of elliptic problems, and probably to many others.

3. Contents of the talks

Yann Brenier (Nice, France):
"L^2-formulation of some hyperbolic conservation laws."

Abstract: It is customary to address hyperbolic conservation laws (or Hamilton-jacobi equations) in functional spaces that are neither Hilbertian nor reflexive (typically L^1, BC, C^0, Lip, etc). We show that, in some simple but significative cases (multidimensional scalar conservation laws, Chaplygin gas etc...), a straightforward L^2 formulation can be introduced, leading to simple well posedness and stability results. This approach can be extended to some degenerate parabolic equations too.

Sun-Yung Alice Chang (Princeton, USA):
"Regularity on a boundary value problem of generalized Yamabe type."

Abstract: In this talk, I will report some recent joint work of Sophie Chen, Paul Yang and myself on a regularity problem of Bach flat metrics on 4-manifolds with boundary. I will discuss both the set up and some ϵ regularity result of the problem.

Fernando Coda Marques (IMPA, Brasil):
"Blow-up analysis for the Yamabe equation in high dimensions."

Abstract: In this talk we will discuss recent progress in understanding how solutions to the Yamabe equation can blow-up in high dimensions. We will describe how to use the Pohozaev identities, under a nondegeneracy condition, to get sufficient vanishing of the Weyl tensor at a blowup point and derive compactness results. This is joint work with Marcus Khuri and Richard Schoen.

Mihalis Dafermos (Cambridge, UK):
"The redshift effect and radiation decay for black hole spacetimes."

Abstract: I shall present new results on decay rates for the wave equation on black hole exterior spacetimes, like Schwarzschild and Schwarzschild-de Sitter, and discuss the relation of this to the problem of non-linear stability of black holes in general relativity. this is joint work with Igor Rodnianski.

Olivier Druet (ENS Lyon, France):
"Quantification of blow-up levels for a 2d elliptic equation with critical exponential growth"

Abstract: We consider sequences of solutions of some 2-dimesnional PDE with critical Trudinger-Moser growthand we show that they split into their weak solution plus a sum of standard bubbles.

Yuxin Ge (Paris 12, France):
"On a conformal quotient equation."

Abstract: In this talk, we consider a conformal quotient equation $\frac{\sigma_2(g)}{\sigma_1(g)} = 1$ in a given conformal class and prove the existence for $n > 4$ and prove a related Sobolev-type inequality.

Nassif Ghoussoub (UBC, Canada):
"On PDEs arising from Electrostatic Micro-Electromechanical Systems"

Abstract: We analyze the nonlinear parabolic problem $u_t = \Delta u - \frac{\lambda f(x)}{(1+u)^2}$ on a bounded domain Ω of $\mathbb{R}^N$ with Dirichlet boundary condition. this equation models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at -1. When a voltage -represented here by λ- is applied, the membrane deflects towards the ground plate and a snap-through may occur when it exceeds a certain critical value $\lambda^\star$ (pull-in voltage). This creates a so-called "pull-in stability" which greatly affects the design of many devices. the challenge is to estimate $\lambda^\star$ in terms of material properties of the membrane, which can be fabricated with a spacially varying dielectric permittivity profile f. Applying analytical and numerical techniques, the existence of $\lambda^\star$ is established together with rigorous bounds. We show the exixtence of at least one steady-state when $\lambda = \lambda^\star$. More refined properties of steady states -such as regularity, stability, uniqueness, multiplkicity, energy estimates and comparison results- are shown to depend on the dimension of the ambient space and on the permittivity profile. As to the dynamic case, the membrane globally converges to its unique maximal negative steady-state when $\lambda \leq \lambda^\star$, with a possibility of touchdown at infinite time when $\lambda = \lambda^\star$. On the other hand, if $\lambda > \lambda^\star$ the membrane must touchdown at finite time T, and

touchdown cannot take place at the location where the permittivity profile vanishes. Both larger pull-in distance and larger pull-in voltage can be achieved by properly tailoring the permittivity profile. We analyze and compare finite touchdown times by applying various analytical and numerical techniques. This is joint work with Yujin Guo.

Hans-Christoph Grunau (Magdeburg, Germany):
"The Paneitz equation in the hyperbolic ball."

Abstract: Existence of a continuum of conformal radially symmetric complete metrics in the hyperbolic ball in $\mathbb{R}^n$, $n > 4$, is shown, all having the same constant Q-curvature. Moreover, similar results can be shown also for suitable non-constant prescribed Q-curvature functions.
(Joint work with M. Ould Ahmedou (Tuebingen), W. Reichel (Aachen))

Colin Guillarmou (Nice, France):
"Generalized Krein formula for Poincaré-Einstein manifolds."

Abstract: We prove a kind of Birman-Krein formula for the Laplacian on even dimensional asymptotically hyperbolic manifolds that have an asymptotic evenness for the metric expansion at infinity. This relates a renormalized trace of the spectral projector of the Laplacian to the phase of the Ray-Singer determinant of the Scattering operator. This is used for instance to obtain a functional equation for Selberg's zeta function on convex co-compact hyperbolic manifolds and to compute the determinant of the GJMS conformal Laplacians in terms of this Selberg's zeta function.

Zheng-Chao Han (Rutgers, USA):
"On the prescribing σ_k curvature equations."

Abstract: Let A_g denote the Schouten-Weyl tensor of a Riemannian metric g on M^n, for $1 \le k \le n$, $\sigma_k(g^{-1} \circ A_g)$ denote the k^{th} elementary symmetric function of the eigenvalues of the $1-1$ tensor $g^{-1} \circ A_g$. For a given function $K(x)$ on M^n, we address several issues in the question of when there exists an admissible conformal metric $g_w = e^{2w(x)} g$ such that

$$\sigma_k(g^{-1} \circ A_g) = K(x) \quad \text{on} \quad M^n. \tag{1}$$

First we will give an elementary and unified discussion to the Kazdan-Warner type necessary conditions for the solvability of (1). Then we will discuss the potential loss of compactness to the solutions of (1), and show that under appropriate non-degeneracy conditions on $K(x)$, so such loss of compactness can happen. This latter part is part of joint work with S.-Y.A. Chang and P. Yang.

Emmanuel Humbert (Nancy, France):
"Surgery and harmonic spinors."

Abstract: Let M be a compact spin manifold with Riemannian metric g. Suppose that N is obtained from M by a surgery of codimension at least 2. We prove that N carries a Riemannian metric h such that the dimension of the kernel of the Dirac operator on (N, h) is not larger than the dimension of the kernel of the Dirac operator on (M, g). As an application, we show that for generic metrics on a spin manifold, the dimension of the kernel of the Dirac operator attains the lower bound given by the index theorem.

Seongtag Kim (Inha, Korea)
Bach-flat manifolds with small L^2-curvature.

Abstarct: In this talk, I will sketch the rigidity of noncompact complete Bach-flat 4-manifolds with small L^2-curvature norm. let (M, g) be a Riemannina four-manifold with Weyl curvature W and Ricci curvature R_{ij}. The BAch tensor B_{ij} is defined by $B_{ij} = \nabla^k\nabla^l W_{kijl} + \frac{1}{2}R^{kl}W_{kijl}$. The Bach tensor is conformally invariant. A manifold (M, g) is called Bach-flat if $B_{ij} = 0$. Important examples of Bach-flat metrics are Eistein metrics, conformally Eistein metrics and self-dual Einstein metrics. We describe geometric conditions that ensure that noncompact complete Bach-flat manifolds with zero scalar curvature and small L^2-curvature norm are flat. For this, we use elliptic estimates on the equation arising from $B_{ij} = 0$.

Andrea Malchiodi (SISSA, Italy):
"Existence of conformal metrics with constant Q-curvature."

Abstract: We discuss the problem of finding conformal metrics on a compact four-manifold which have constant Q-curvature. This consists in solving a fourth order elliptic PDE with exponential nonlinearity and with variational structure. When the total Q-curvature is large, the Euler functional is unbounded from below, and critical points have to be found via min-max methods. Using a new variational scheme based on concentration of conformal volume, we solve the problem for a large class of manifolds.

Rafe Mazzeo (Stanford, USA):
"Geometric foliations near infinity in asymptotically hyperbolic manifolds."

Abstract: I will describe recent work with Frank Pacard concerning the existence and uniqueness of foliations near infinity in asymptotically hyperbolic spaces such that each leaf has constant mean (or σ_k) curvature, and the relationship of these foliations to the conformal infinity of these spaces.

Robert McCann (Toronto, Canada):
"Regularity and counterexamples in optimal transportation."

Abstract: I shall give a rapid survey of new and old results in optimal transportation, including the regularity theory of Ma, Trudinger, Wang and Loeper, and counterexamples to regularity due to Gregoire Loeper.

Frank Pacard (Paris 12, France):
"Elliptic aspects of extremal Kähler metrics."

Abstract: I will report some recent results with C. Arezzo and M. Singer about the existence of extremal Kähler metrics on the blow up at finitely many points of manifolds wich carry an extremal Kähler metric.

Jie Qing (Santa Cruz, USA): tba.
"Metrics degeneration and its applications in conformal geometry." Abstract: In this talk, we develop a bubble tree structure for a degenerating class of Riemannian metrics satisfying some global conformal bounds on compact manifolds of dimension 4. Applying the bubble tree structure, we establish a gap theorem, a finiteness theorem for diffeomrphism type for this class, and raise a question concerning a

diameter bound for a family of coformal metrics satisfyning a suitable curvature equation.

Michael Struwe (ETH Zürich, Switzerland):
"Partial regularity for harmonic maps, revisited."

Abstract: Via gauge theory, we give a new proof of partial regularity for harmonic maps in dimensions $m \geq 3$ into arbitrary targets. This proof avoids the use of adapted frames and permits to consider targets of "minimal" C^2 regularity. The proof we present moreover extends to a large class of elliptic systems of quadratic growth.

Robert Van der Vorst (Amsterdam, Holland):
"Closed characteristics on non-compact hypersurfaces."

Abstract: C.Viterbo proved that any $(2n-1)$-dimensional compact hypersurface $M \subset (\mathbb{R}^{2n}, \omega)$ of contact type has at least one closed characteristic. This result proved the Weinstein conjecture for the standard symplectic space $(\mathbb{R}^{2n}, \omega)$. Various extensions of this theorem have been proved since, all for compact hypersurfaces. In this paper we consider *non-compact* hypersurfaces $M \subset (\mathbb{R}^{2n}, \omega)$ coming from mechanical Hamiltonians, and prove an analogue of Viterbo's result. The main result provides a strong connection between the homology groups $H_k(M)$, $k = n, \cdots, 2n-1$, and the existence of closed characteristics in the non-compact case (including the compact case).

Paul Yang (Princeton, USA)
Title: *"Minimal surfaces in pseudo-Hermitian geometry."*

Abstract: I will discuss the notion of minimal surfaces in a pseudo-Hermitian manifold, the existence and uniqueness questions as well as that of the regularity. It turns out that in dimension three, there are serious regularity questions. These are joint works with J.Cheng, J.Huang and A.Malchiodi. I will also discuss the Sobolev inequality on a pseudo-Hermitian manifold, and report on some joint work with S.Chanillo.