Topics on von Neumann algebras

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September 16–21, 2006

The emphasis of the workshop was given to active areas in the theory of von Neumann algebras with connections to other fields as well as to these areas themselves. Among the participants were almost all the leaders in von Neumann algebras, but also representatives from group theory, quantum computing and conformal field theory. As well, in attendance were one Fields medalist (V. Jones), two speakers who gave plenary talks at an ICM (U. Haagerup, 2002, and S. Popa, 2006), and five additional mathematicians who have given invited addresses at ICMs, with two of them at the recent one in 2006 (N. Ozawa and N. Monod). Besides these and other well-established experts in von Neumann algebras and related fields, there were also many younger participants (including recent Ph.D.'s, graduate students, and a good representation of women) who had the opportunity to interact with these leaders. Participants expressed in numerous occasions that the workshop was very stimulating and allowed for fruitful discussions of joint projects. We are glad that the objectives for this workshop were fulfilled.

1 Short overview of the field and topics targeted in the workshop

Von Neumann algebras are algebras of bounded linear operators on a Hilbert space which are closed under the topology of pointwise convergence. If their center only consists of multiples of the identity, they are called factors. Von Neumann algebras were first studied in a series of papers by Murray and von Neumann in the 1930's, such as in [MvN]. Their motivation was to have a tool for studying quantum mechanics and representations of infinite groups. As it will be seen below, these are still some of the major driving forces in research related to von Neumann algebras with exciting recent developments.

In order to better describe the structure of the workshop scientific content –in which the allotted time was separated into thematic groupings– we will roughly divide the recent activities in von Neumann algebras into the study of subfactors and the study of factors.

1. *Subfactors.* The study of subfactors was initiated by V. Jones in the 1980's by introducing an important invariant for them called the index, [J1]. Moreover, he proved a surprising and fundamental theorem on the set of possible index values and he produced an important class of examples called the Jones subfactors. This class of examples carried a representation of braid groups, and was later used to define link invariants, [J2]. This in turn led to invariants of 3-manifolds and to connections to conformal field theories, representations of loop groups and quantum groups and fusion categories (see e.g. [Wi], [RT], [Wa] and [We]). Some of these connections will be discussed below.

An important classification result for amenable subfactors of the hyperfinite II_1 factor was proved by S. Popa, [P1]. He showed that they can be reconstructed by what he calls the standard invariant; it is, however, still a wide open problem what values this standard invariant can take in general. The following topics

are still very active areas in connection with subfactors, which were addressed in the workshop.

(a) *Fusion*. A. Wassermann's construction [Wa] was in part inspired by results in algebraic quantum field theory, where von Neumann algebras have played an important role for a long time. His crucial result in this context was the definition and explicit computation of a highly nontrivial tensor product between two representations of loop groups of the same level, which is usually called the fusion tensor product. A. Wassermann's fusion can be considered a limiting case of a fusion proposed by G. Segal in connection with conformal field theory. The latter fusion still has not been mathematically rigorously established. Wassermann is currently working on these problems, which he has discussed in his talk in the workshop. In a related development, relative tensor products of bimodules are also investigated by Jones, who gave a talk on his current progress at this workshop as well.

(b) *Conformal Field Theory and Subfactors*. As already mentioned, Wassermann's work established a useful link between conformal field theory and von Neumann algebras: As an application, he obtained examples of subfactors, which could also be constructed on the type II_1 level via quantum groups [We],[Xu1]. However, the conformal field theory approach also yields additional constructions which, at least so far, can not be done via technically less demanding approaches. More progress in the construction of new examples has been achieved in recent works by F. Xu by his own, [Xu2], and also in joint work with V. Kac and R. Longo, [KLX]. Additional constructions are to be expected from the machinery of conformal field theory. Moreover, various aspects of the connections between the theory of von Neumann algebras and algebraic as well as conformal field theory also appeared in talks given by D. Evans, T. Gannon, Y. Kawahigashi, and R. Longo (see 'presentation highlights').

(c) *Other constructions.* There are also constructions of subfactors via other methods. One of these yields the famous Haagerup subfactor, the irreducible subfactor of the hyperfinite II_1 factor of smallest known index > 4, [AH]. This subfactor was obtained from a list of possible standard invariants provided by Haagerup, [H]. There has been recent progress in showing that these other candidates for possible standard invariants probably do not give subfactors, by M. Asaeda [A], about which she gave a talk in this Banff workshop. Her proof uses results by T. Gannon, [CG], coming from his algebraic/combinatorial studies of conformal field theory related topics, and by P. Etingof, D. Nikshych and V. Ostrik, [ENO].

(d) *Further examples/counterexamples.* Popa's recent results (see part 2) have also had consequences for the study of subfactors. In particular, D. Bisch, R. Nicoara, and Popa, [BNP], have recently constructed a continuous family of mutually non-isomorphic irreducible finite-index subfactors of the hyperfinite type II_1 factor with the same standard invariant, about which Bisch gave a talk at this workshop. Also S. Vaes recently showed the existence of type II_1 factors whose only finite-index subfactors are the trivial ones, [Va], using techniques by Popa, A. Ioana and J. Peterson (of the type described in 2.(b)); a talk about these techniques was given at this workshop as well.

(e) Certain tensor categories connected to subfactors and conformal field also play an important role in Freedman's approach towards building a quantum computer. A survey on that was given by E. Rowell.

2. *Factors and free probability.* One of the big problems in von Neumann algebras is the classification of II_1 factors. One can define an important class of examples of such factors from the group von Neumann algebras of infinite discrete groups for which all nontrivial conjugate classes are infinite. However, it is very difficult to decide when these factors are isomorphic. Various invariants for II_1 factors have been introduced by A. Connes, e.g. [Co], and several deep results have been proved by him. He showed all the factors obtained from amenable groups are isomorphic to the hyperfinite II_1 factor. It is known that this factor is not isomorphic to the one obtained from a free group with n generators.

(a) It has been a longstanding unsolved problem to decide whether the factors obtained from the free groups with n and m generators respectively are isomorphic if n is not equal to m with both $n, m > 1$. This problem was one of the inspirations for Voiculescu's theory of free probability. While this has not led to a solution of the original problem yet, it produced many interesting results in its own right as well as surprising

applications to the theory of von Neumann algebras such as e.g. Voiculescu's proof of the absence of Cartan subalgebras for free group factors, [Vo]. Voiculescu and his school have found amazing analogs in free probability of well-known phenomena in classical probability, such as e.g. free Fisher information and free entropy. Questions involving these concepts have been discussed in talks in this Banff workshop (see the talks by D. Shlyakhtenko, K. Dykema and K. Jung).

(b) The most exciting developments in the theory of von Neumann algebras in the last few years undoubtedly took place in connection with group theory. D. Gaboriau defined a notion of ℓ^2 Betti numbers for countable measure preserving equivalence relations in a Borel space, [Ga]. This proved a crucial tool in Popa's proof of a long-standing problem in von Neumann algebras, the construction of a II_1 factor with trivial fundamental group, [P2]. In addition Popa has continued proving exciting (super)rigidity results concerning group actions on probability spaces. More precisely, he shows for certain groups acting on probability spaces that an equivalence between their orbits already induces an equivalence between the groups themselves, e.g. [P3]. These results, among others, strongly contributed to the fact that Popa was an invited plenary speaker at the ICM 2006.

There have been similar superrigidity results within geometric group theory by Y. Shalom, N. Monod and A. Furman, with the first two researchers also honored at the recent ICM. From our 2006 workshop, N. Monod and N. Ozawa expect to collaborate on the study of von Neumann algebras associated to arithmetic groups. This is a very active area, with interesting new results also coming from young people such as A. Ioana and J. Peterson who also spoke at our workshop.

Additional interesting results were recently obtained by N. Ozawa: Based on his notion of solid von Neumann algebras, [O1], he obtained many examples of prime factors (i.e. II_1 factors which are not the tensor product of two II_1 factors) [O2]; moreover, in collaboration with Popa, [OP], they prove unique prime factorization results for tensor products of factors coming from subgroups of hyperbolic groups. New results in this direction have also been presented in this workshop by Peterson.

(c) In another interesting development Haagerup talked about his recent work with M. Musat on classification of hyperfinite factors up to completely bounded isomorphism of their preduals. It follows from their work that they can distinguish between various hyperfinite III_0 factors. This is an important class of factors which still are not very well understood. So this is an important result whose details have not appeared yet. The Banff workshop turned to be useful for both Haagerup and Musat to continue their collaboration.

2 Presentation Highlights (following same thematic order as in the workshop schedule)

Day 1.

Sorin Popa (University of California, Los Angeles) *On the Superrigidity of Malleable Actions* (Abstract): Let $\Gamma \curvearrowright X$ be a measure preserving action of a countable discrete group on a probability space. It is well understood by now that some weak form of property (T) for Γ combined with a *malleability* assumption on the way it acts on X entails sharp rigidity phenomena for the associated II_1 factor and orbit equivalence relation. I will present a new set of rigidity results for malleable actions, in which the property (T) assumption on Γ is no longer needed. Instead, the group needs to have a non-amenable subgroup H with infinite centralizer.

Jesse Peterson (University of California, Berkeley) *L^2-rigidity in von Neumann algebras* (Abstract): I will present a new approach for showing primeness in von Neumann algebras. Specifically I will apply Popa's deformation/rigidity techniques in the context of Sauvageot's deformations arising from closable derivations to conclude that all free product II_1 factors, as well as all group factors arising from groups with positive first L^2-Betti number are prime. These techniques also give a new approach to Ozawa's result that all nonamenable subfactors of a free group factor are prime.

Narutaka Ozawa (University of Tokyo) *A comment on the free group factors* (Abstract): For a finite von Neumann algebra M, there are natural inclusions $M \subset L^2 \subset L^1$. I will talk about the space of those operators in $B(L^2)$ that are compact when viewed as operators from M into L^p ($p = 2, 1$). I will particularly discuss the free group factors.

Nicolas Monod (University of Geneva) *Splitting and rigidity* (Abstract): The study of Orbit Equivalence for type II_1 relations is a classical topic in ergodic theory. It is known to be equivalent to the von Neumann algebra setting of Cartan subalgebras of factors of type II_1. We start by presenting results from joint work with Shalom in which strong restrictions, and indeed superrigidity, are established for relations produced by suitable product groups. These results hinge upon bounded cohomology, in particular upon a splitting theorem in the latter formalism. We then briefly present a set of results bearing a formal analogy with the preceding: A general version of Margulis' superrigidity for latices in products, which relies on a geometric splitting theorem generalising the classical Lawson-Yau/Gromoll-Wolf theorem for Hadamard manifolds.

Mikaël Pichot (Institut des Hautes Etudes Scientifiques (IHES)) *The space of triangle buildings* (Abstract): I will present the notion of measured equivalence relation from a geometric point of view and discuss the concentration of measure property in that framework. Examples of measured equivalence relations arise, for instance, from nonsingular actions of groups on probability spaces, foliation theory, questions of classification,.... A part of the talk will be devoted to the problem of classification of triangle buildings (joint work with Sylvain Barre).

Adrian Ioana (University of California, Los Angeles) *Orbit inequivalent actions for groups containing a copy of* F_2.

Day 2.

Vaughan Jones (University of California, Berkeley) *Connes tensor product in quantum physics?* (Abstract): We present evidence for treating highly constrained physical systems using the Connes tensor product of correspondences. The idea is that the constraints should force some observables on one system to be identified with observables on the other system.

Pinhas Grossman (Vanderbilt University) *Forked Temperley-Lieb Algebras and Intermediate Subfactors* (Abstract): We consider noncommuting pairs P, Q of intermediate subfactors of an irreducible, finite-index inclusion N in M of II_1 factors such that P and Q are supertransitive with Jones index less than 4 over N. We show that in the hyperfinite case, there is a unique such pair corresponding to each even value $[P : N] = 4\cos^2(\pi/2n)$ but none for the odd values $[P : N] = 4\cos^2(\pi/(2n+1))$.

Antony Wassermann (CNRS, Institut de Mathématiques, Luminy) *Segal and Connes' fusion*: In this talk an important technical problem for a rigorous definition of Segal fusion is dicussed, fusion with corners. This leads to classical questions in connection with uniformization.

V.S. Sunder (IMSc, Chennai) *Kac algebras, doubles and planar algebras* (Abstract): We wish to describe the planar algebra of the 'double' (=asymptotic inclusion) of the fixed subfactor $R^H \subset R$ of an outer action of a finite-dimensional Kac algebra on the hyperfinite factor. It turns out that this happens to be identifiable as a sub-planar algebra of the related subfactor $R^{H^{*op}} \subset R$. We describe this result, some corollaries and an idea of the ingredients of the proof.

Dietmar Bisch (Vanderbilt University) *A continuous family of hyperfinite subfactors* (Abstract): I will present a construction of continuous families of non-isomorphic, irreducible, finite index subfactors of the hyperfinite II_1 factor with the same standard invariant. This is joint work with Remus Nicoara and Sorin Popa.

Masaki Izumi (Kyoto University) *Type III factors distinguish (some) type III E_0-semigroups* (Abstract): I'll give an account of new examples of uncountably many type III E_0-semigroups, which are distinguished by the type of analogues of the local observable algebras. Joint work with R. Srinivasan.

Eric Rowell (Purdue University) *Algebraic Problems in Topological Quantum Computing*: A brief overview was given about the Freedman approach towards building quantum computers. This motivated various algebraic and combinatorial questions related to unitary tensor categories, some of which were discussed in more detail.

Stefaan Vaes (Institut de Mathématiques de Jussieu) *Type II_1 factors without non-trivial finite index subfactors* (Abstract): We call a subfactor trivial if it is isomorphic with the diagonal inclusion of N into matrices over N. We prove the existence of type II_1 factors M such that every finite index subfactor is trivial. Also, every M-M-bimodule with finite coupling constant, is a multiple of $L^2(M)$. In particular, these II_1 factors do not have outer automorphisms: such factors were shown to exist by Ioana, Peterson, Popa and our methods are a generalization of theirs.

Day 3.

Uffe Haagerup (University of Southern Denmark) *Classification of hyperfinite factors up to completely bounded isomorphism of their preduals (joint work with Magdalena Musat)* (Abstract): By a result of Christensen and Sinclair, all infinite dimensional hyperfinite factors are cb-isomorphic (i.e. isomorphic as operator spaces), but if one looks at the preduals instead, the story is totally different: It turns out, that one can for instance separate Type II from Type III this way, and there are uncountably many non-cb-isomorphic preduals of hyperfinite Type III_0 factors, while the preduals of hyperfinite Type III_λ factors are all isomorphic when $0 < \lambda \leq 1$. The proof uses Connes classification of injective (=hyperfinite) factors and the Connes-Takesaki "flow of weights" for Type III-factors.

Kenley Jung (University of California, Los Angeles) *Microstate Spaces and Geometric Measure Theory in Free Probability* (Abstract): I will discuss the microstate theory in free probability and its operator algebra applications. Emphasis will be placed on the geometric measure theory approach to microstates and on how this alternative interpretation leads to nonisomorphism results for von Neumann algebras.

Dmitri Shlyakhtenko (University of California, Los Angeles) *Estimates for free entropy dimension* (Abstract): We discuss some new and old estimates on free entropy dimension and connections with a free analog of an inequality of Otto and Villani (previously considered in the free case by Biane and Voiculescu) that occurs in their work on the Talagrand inequality.

Ken Dykema (Texas A&M University) *Free Entropy Dimension in Amalgamated Free Products* (Abstract): We calculate the free entropy dimension of natural generators in an amalgamated free product of the hyperfinite II_1-factor with itself, with amalgamation over an atomic, type I subalgebra. In particular, some 'exotic' Popa algebra generators of free group factors are shown to have the expected free entropy dimension. (Joint work with Nate Brown and Kenley Jung.)

Day 4.

Terry Gannon (University of Alberta) *The braid group and modular forms (among other things)*: In this talk it is argued that for various problems in conformal field theory the group $SL(2, Z)$ should be replaced by the 3-strand braid group.

Marta Asaeda (University of California, Riverside) *Galois group obstruction to principal graphs* (Quoted from math.OA/0605318): The Galois group of the minimal polymonal of a Jones index value gives a new type of obstruction to a principal graph, thanks to a recent result of P. Etingof, D. Nikshych, and V.Ostrik. We show that the sequence of the graphs given by Haagerup as candidates of principal graphs of subfactors, are not realized as principal graphs for $7 < n \leq 27$ using GAP program. We further utilize Mathematica to extend the statement to $27 < n \leq 55$. We conjecture that none of the graphs are principal graphs for all $n > 7$, and give evidence using Mathematica for smaller graphs among them for $n > 55$. The problem for the case $n = 7$ remains open, however, it is highly likely that it would be realized as a principal graph, thanks to numerical computation by Ikeda.

David Evans (Cardiff University) *Modular invariants, Subfactors and Twisted K-theory*.

Shamindra Ghosh (Vanderbilt University) *Planar algebras (Abstract): A category theoretic point of view*: We define Jones's planar algebra as a map of multicategories and construct a planar algebra starting from a 1-cell in a pivotal strict 2-category. We introduce the concept of an affine representations of a planar algebra and prove some finiteness results for the affine representations of finite depth planar algebras. We also show that the radius of convergence of the dimension of an affine representation of the planar algebra associated to a finite depth subfactor is at least as big as the inverse-square of the modulus.

Yasuyuki Kawahigashi (University of Tokyo) *Superconformal nets of factors and their classification* (Abstract): Super Virasoro nets with central charge less than $3/2$ are constructed as Fermionic extensions of certain coset nets arising from the $SU(2)$-nets, as studied by A. Wassermann and F. Xu. This construction is an operator algebraic counterpart for the Goodard-Kent-Olive coset construction for the discrete series for the $N = 1$ superconformal algebras. We study their extensions, and give a complete classification, using the work on modular invariants by Cappelli and Gannon-Walton. This is a "super" counterpart of our previous complete classification of local conformal nets with central charge less than 1. This is a joint work with Roberto Longo.

Roberto Longo (University of Rome Tor Vergata) *Nuclearity for inclusions of real Hilbert spaces, representations of $SL(2, R)$ and CFT* (Abstract): (Based on a joint work with C. D'Antoni and D. Buchholz) We introduce a new type of spectral density condition, that we call L^2-nuclearity. One formulation concerns lowest weight unitary representations of $\mathrm{SL}(2, R)$ and turns out to be equivalent to the existence of characters. A second formulation concerns inclusions standard real Hilbert subspaces of a complex Hilbert spaces. We consider Moebius covariant nets of real Hilbert subspaces associated with interval on the circle and set up a relation with the above nuclearity conditions. We show the corresponding nuclearity conditions to agree for a local conformal net of von Neumann algebras on the circle (chiral conformal Quantum Field Theory) and, starting from the trace class condition for the semigroup generated by the conformal Hamiltonian L_0, we infer and naturally estimate the Buchholz-Wichmann nuclearity condition and the (distal) split property. As a corollary, if L_0 is log-elliptic, the Buchholz-Junglas set up is realized and so there exists a beta-KMS state for the translation dynamics on the net of C^*-algebras for every inverse temperature beta> 0. We further mention a formulation on higher dimensional spacetimes. In particular, L^2-nuclearity is satisfied for the scalar, massless Klein-Gordon field.

Feng Xu (University of California, Riverside) *Mirror extensions of local nets* (Abstract): In this talk we will discuss a general theorem which under certain conditions constructs extensions of local nets from given ones. Such extensions are called mirror extensions since the corresponding link invariants are related to their mirror images in the given nets. When applying the theorem to conformal inclusions and diagonal cosets, we obtain infinite series of new examples of completely rational chiral conformal field theories. The talk is based on math.QA/0505367.

3 Final comments

As mentioned at the beginning, the workshop was very successful in bringing together almost all the leading experts in von Neumann algebras as well as researchers from related areas. We think it provided an excellent reflection of the current exciting developments in this subject and its influences on/from other areas. This should be particularly helpful for the many younger researchers which attended our workshop.

We received positive comments and feedback about the meeting from many participants. So we believe that it was indeed stimulating and did contribute to further progress in our field.

4 Participants

Argerami, Martin (University of Regina)
Asaeda, Marta (University of California, Riverside)
Bisch, Dietmar (Vanderbilt University)

Ciuperca, Alin (University of Toronto)
Dykema, Ken (Texas A&M University)
Elliott, George (University of Toronto)
Erlijman, Juliana (University of Regina)
Evans, David (Cardiff University)
Gannon, Terry (University of Alberta)
Ghosh, Shamindra (Vanderbilt University)
Goodman, Fred (The University of Iowa)
Grossman, Pinhas (Vanderbilt University)
Haagerup, Uffe (University of Southern Denmark)
Hauschild Mosley, Holly (Grinnell College)
Ioana, Adrian (Uiversity of California, Los Angeles)
Izumi, Masaki (Kyoto University)
Jones, Vaughaun (University of California, Berkeley)
Jung, Kenley (University of California, Los Angeles)
Kawahigashi, Yasuyuki (University of Tokyo)
Longo, Roberto (University of Rome Tor Vergata)
Massey, Pedro (Universidad de la Plata)
Monod, Nicolas (University of Geneva)
Musat, Magdalena (University of Memphis)
Niu, Zhuang (University of Calgary)
Ozawa, Narukata (University of Tokyo)
Peterson, Jesse (University of California, Berkeley)
Pichot, Mikaël (Institut des Hautes Etudes Scientifiques (IHES))
Popa, Sorin (university of California, Los Angeles)
Robert, Leonel (University of Toronto)
Rowell, Eric (Texas A&M University)
Santiago Moreno, Luis (University of Toronto)
Sasyk, Román (University of Purdue)
Shlyakthenko, Dimitri (University of California, Los Angeles)
Sunder, V.S. (The Institute of Mathematical Sciences, India)
Tuba, Imre (San Diego State University, Imperial Valley Campus)
Vaes, Stefaan (Institut de Mathématiques de Jussieu)
Viola, Maria Grazia (Queen's University)
Wassermann, Antony (CNRS, Institut de Matheématiques, Luminy)
Wenzl, Hans (University of California, San Diego)
Xu, Feng (University of California, Riverside)

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