Nonlinear diffusions: entropies, asymptotic behavior and applications

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1 Overview of the Field

Scientists have long recognized the importance of diffusion in modelling everything from stock market prices to the spread of environmental pollutants. If feedback mechanisms are present in the system, then the rates of the process may vary from point to point depending on the gradient and concentration of the material diffusing. The system may then exhibit dramatic transitions, nonlinear behaviour, and pattern formation which challenge prediction and analysis. Such models are used in the manufacture of semiconductors, oil recovery, epidemiology of diseases, and the assessments of environmental impact. There is a strong connection between analytical progress in diffusion problems and their applications in the physical, biological, and engineering sciences. As an example we point out the developments in semiconductor drift-diffusion modelling of the last 30 years, which to a great extent was co-responsible for the successful design of many generations of very large scale integrated (VLSI) structures.

Nonlinear diffusion also plays a key role in other physical, biological, and geometric processes, such as fluid seepage, population spreading, pattern formation, chemotaxis, reaction dynamics, curvature flows, thermalization in plasmas, and avalanches in sandpiles. Such processes are modelled using partial differential equations (PDE), whose stability, structure, and large time dynamics are questions of great relevance to understanding the qualitative and quantitative behaviour of the models and the processes they govern. The same differential equations have also emerged at the heart of Perelman and Hamilton's proposed solution to some of the deepest problems in geometry and topology, which include enumerating the ways in which a three-dimensional universe may connect with itself. This workshop brought leading experts together with a younger generation of aspiring and accomplished scientists to explore the state of the art in nonlinear diffusion, and to set the agenda concerning the mathematical and modelling challenges in this vital area of nonlinear PDE.

Many of the equations discussed at this meeting had a variational form

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(m(u)\,\operatorname{grad}\frac{\delta\mathcal{H}(u)}{\delta u}\right) , \tag{1}$$

where $u(t,x) \geq 0$ is the density of material diffusing, $m : [0,\infty[\longrightarrow [0,\infty[$ determines its *mobility* or response to forcing by pressure, and the strength of the pressure is determined by a *Lyapunov functional* $\mathcal{H}(u)$, which gives a dissipated *energy* or *entropy*. In the simplest, zeroth order-case, $\mathcal{H}(u)$ depends only on the value of $u(x)$, and not on its derivatives. The resulting evolution, like the heat equation, enjoys a comparison principle which is a key tool in analyzing the solutions, though the nonlinearity may still cause vexing problems. If $\mathcal{H}(u)$ depends also on the gradient of u (i.e., is a first order functional), the fourth-order evolution (1) which results has no maximum principle and poses an entirely different set of analytical challenges which have only quite recently begun to be explored. Since first-order energies arise in the modeling of semiconductors, thin-films, and statistical mechanical fluctuations, they have both commercial and academic interest.

The fundamental question is to develop an understanding of how the subsequent evolution depends on the initial conditions and boundary data. This understanding can be based on explicit solutions — usually very rare and depending strongly on separation of variables and symmetry in the data and boundary conditions — perturbation theory and asymptotic analysis, local and global inequalities such as a priori estimates, comparison theorems, numerical simulations and laboratory experiments, and finding the right functional spaces and settings to obtain well-posedness results.

Points of contact between diffusion equations, and kinetic type equations, like the spatially homogeneous Boltzmann equation

$$\frac{\partial u}{\partial t} = \mathcal{Q}(u) \ , \tag{2}$$

were also discussed. Both types of equations are dissipative. Indeed, a simple formal calculation shows that for a solution $u(t,x)$ of (1), the energy $\mathcal{H}(u(t))$ decreases in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(u) = -\int m(u) \left| \mathrm{grad} \frac{\delta \mathcal{H}(u)}{\delta u} \right|^2 \mathrm{d}x. \tag{3}$$

Of course, one of the oldest results in this direction is Boltzmann's H–Theorem, which asserts dissipativity of the relative entropy with respect to equilibrium for solutions of (2). That is, let u be a solution of (2) whose initial data is a probability density with a finite second moment. Let M be the Maxwellian density (i.e., Gaussian with isotropic covariance) that has the same first and second moments. The relative entropy of the solution with respect to the equilibrium solution M is

$$\int \ln \left(\frac{u(v,t)}{M(v)} \right) u(v,t) \mathrm{d}v \tag{4}$$

where v is often used instead of x for the argument for u because these variables represent particle velocities in the kinetic context. Boltzmann's H–Theorem asserts that this quantity is monotone decreasing along the evolution described by (2). It strongly suggests — and in some cases can be used to prove — that all initial data in a suitable class will become more and more Gaussianly-shaped as time progresses.

Still other types of equations were discussed at the meeting, as we shall explain, but let us start here to keep our report concrete. The issues discussed with regard to other PDE can often be related to issues which pertain also to (1) and to (2).

2 Recent Developments and Open Problems

The last few years have witnessed remarkable progress on this subject. Despite the antiquity of the H–Theorem, the use of relative entropy inequalities to obtain *a priori* information on the *rate* of convergence to equilibrium is relatively recent. The previous generation of researchers in the subject relied on spectral estimates coming from a linearization that were valid only in a very small neighborhood of the equilibrium.

Entropy dissipation arguments have recently been employed to study nonlinear diffusion by a number of researchers. An important part of much recent progress has been based on a a geometrical point of view on these evolutions introduced by Felix Otto, who highlighted deep connections between nonlinear diffusion and the variational problems of mass-transport theory.

In particular, in the case that the mobility function m in (1) is just the identity $m(u) = u$, the equation (1) is the gradient flow for the functional $\mathcal{H}$ with respect to the 2–Wasserstein metric coming from mass transport theory. In other words, the dynamics (1) is the steepest descent of the energy $\mathcal{H}$, provided the distance between two probability densities is taken to be the square of the average spatial distance required to displace the particles of one to the other. Otto showed that that when $\mathcal{H}$ is a strictly convex function on the "Riemannian manifold" of probability densities, where the Riemanian metric is induced by the 2–Wasserstein metric, then quantitative measures of the strict convexity of $\mathcal{H}$ had quantitative implications for both the rate of convergence problem, and the contractivity problem for solutions of (1). Mass transport theory had recently undergone significant development, building on the work of Brenier and McCann in particular, and this provided many new tools, from an unexpected source, to the study of PDE such as (1).

Otto applied this geometric perspective to the porous medium equation, in which case one has

$$\mathcal{H}(u) = \frac{1}{p}\int u^p(x)dx \tag{5}$$

for $p > 1$. The point is that the strict convexity of $\mathcal{H}$ implies an inequality between $\mathcal{H}$ itself, and its rate of dissipation. Such "entropy–entropy dissipation" inequalites are generally very hard to prove by means of a direct comparrison between $\mathcal{H}$ and the dissipation rate given in (3). The geometric framework introduced by Otto has provided a powerful new perspective on how to prove such inequalities.

If one takes $\mathcal{H}$ to be the relative entropy as given by (4), then (1) becomes the Fokker–Planck equation with M as the equilibrium solution. In this case, the "entropy–entropy dissipation" inequality was already well known; it is the celebrated logarithmic Sobolev ineuqality of Gross. Much progress has since been made in applying mass transport techniques to the study of logarithmic sobolev inequalities in other settings, and to proving other functional inequalities; e.g., Gagliardo–Nirenberg type inequalities.

A fruitful interaction between the mass transport and the PDE communities has grown rapidly since then, mainly due to the interchange of points of view, different techniques, and progress on the open problems in both subjects. New ideas have led to many improvements in the mathematical results concerning rates of convergence, functional inequalities, sharp geometric constants, and links to optimal mass transportation and kinetic theories. Perelman's announced proof of the geometrization conjecture is a spectacular application of nonlinear diffusion (Ricci flow) to fundamental problems in geometry and topology.

The understanding of large time asymptotics is best developed for model problems such as the porous medium / fast diffusion equation and related nonlinear Fokker-Planck equations, which were the initial objects of investigation. However, even these simple models pose open questions concerning rates of decay for restricted initial data, higher order asymptotics, estimates on the location and evolution of free boundaries, perturbations by convex potentials. The adaptation of relative entropy or variational techniques to other PDEs and coupled systems displaying more complicated dynamics is largely a challenge for the future, though very interesting progress has been recently made, and was the subject of much discussion at the meeting. For example, while the porous medium equation and the Fokker–Planck equation arise through the use of zeroth order functionals $\mathcal{H}(u)$, such as those in (4) and (5), a much more challenging set of problems arise when one considers first order functionals such as

$$\mathcal{H}(u) = \int |\nabla u|^2 \mathrm{d}x \qquad \text{or} \qquad \mathcal{H}(u) = \int \frac{|\nabla u|^2}{u}\mathrm{d}x\ .$$

These give rise, respectively to the thin film equation, and an equation known as the Derida–Lebowitz–Speer–Spohn (DLSS) equation. These are fourth order equations, and hence there is no maximum principle argument to ensure that solutions with non negative initial data stay non negative. Whether this is even true of not depends on the mobility $m(u)$: For the thin film equation, it is known that if this is degenerate enough at $u = 0$, then the evolution "slows down" as when a solution approaches zero, and positivity is maintained. However, for a mobility of the form $m(u) = u^p$, the smallest value of p for which positivity of the initial data implies positivity of solutions is unknown. This open problem was much discussed at the meeting.

The fact that one can view the DLSS equation as coming from gradient flow with respect to the 2–Wasserstein metric does, however, at least provide a way to construct non–negative solutions. There are many technical difficulties, though, in working with these higher order functionals. The most significant is that the are not convex with respect to the 2–Wasserstein metric, and so the contraction property that hold in the presence of convexity cannot be invoked to assert the uniqueness of the solutions constructed by the gradient flow argument. It remains an open problem to deal with this difficulty.

What is known about the positivity of solutions of the thin film equation comes from the analysis of a family of Lyapunov functionals discovered by Bernis and Friedman. Not only is $\mathcal{H}(u) = \int |\nabla u|^2 \mathrm{d}x$

dissipated under the evolution described by the thin film equation, but so are other functions: The ones found by Bernis and Friedman are zeroth order, and other first order functionals have been found by Laugesen. This raises the question: Given a dissipative evolution equation such as (1), is there a systematic way to find other Lyapunov functionals (besides $\mathcal{H}(u)$) for its evolution? A framwork for doing this was presented by A. Jungel and D. Matthes. It resulted in much discussion at the meeting, with specific problems to which the method might be applied being identified. However, other natural question seem to stymie this method at present. For example, one would like to know whether there are functionals involving a term of the form

$$\int \frac{|u_{xx}|^2}{u^p} \mathrm{d}x$$

with some $p > 0$ that are monotone decreasing for the thin film evolution. The functionals found by Laugessen have the form

$$\int \frac{|u_x|^2}{u^p} \mathrm{d}x \ .$$

A better understanding of where they come from, and how one might find others, was the subject of much discussion.

Another focus of the meeting was flux-limited diffusion equations in radiation hydrodynamics. These are interesting since they are the source of new developments in the analysis of bounded variation functions. A typical example here is

$$u_t = \nu \,\mathrm{div} \left(\frac{uDu}{\sqrt{u^2 + \frac{\nu^2}{c^2}|Du|^2}} \right) , \tag{6}$$

known as the *relativistic heat equation*. This is of a somewhat different form than (1), but it can still be derived in the same way as the other $m(u) = u$ examples of (1) by means of the Monge-Kantorovich mass transport theory. This was shown by Brenier, who used a cost function that stipulated an infinite penalty for transporting mass too far. Brenier's work was done at a formal level, and much discussion at the conference focused on how to rigorously and numerically handle such equations.

As evidenced at this meeting, there is a healthy interaction between new developments in mass transport theory and the classical tools of PDE, such as self-similarity, maximum principles, L^p contractivity, compactness methods, smoothing effects, dynamical systems arguments. Moreover, the new techniques mentioned above combine with more traditional approaches to impact questions in kinetic theory such as thermal equilibration rates for homogeneous gases, and well-posedness of inhomogeneous models by variational schemes. Several metrics connected to Wasserstein distances have been shown to describe the asymptotics in spatially-homogeneous kinetic problems.

Indeed, an old result of Tanaka is that the evolution of the spatially homogeneous Boltzmann equation (2) for so–called "Maxwellian molecules" is contractive is the 2–Wasserstein metric – just as if (2) were the gradient flow for some convex functional with respect to the 2–Wasserstein metric. A very interesting open problem discussed at the meeting is whether this contraction property, perhaps with respect to a mass–transport metric based on some other cost functional, can be extended to (2) in general, not just for Maxwellian molecules. Good evidence for the robustness of Tanaka's result, and the plausibility of this conjecture, was given in Carrillo's talk, where he extended Tanaka's result to the case of dissipative collisions, such as arise in granular flows.

3 Presentation Highlights

Presentations at this meeting ranged focused mainly on problems in mathematical analysis, though a smaller number of talks were devoted to computational and theoretical modeling and phenomenology.

Connections between nonlinear diffusion and geometric (Ricci) curvature flows were a frequent theme. State of the art results were reviewed by Daskalopoulos, Ni and Vázquez on this subject. Entropy-type arguments, Li-Yau-Hamilton inequalities, existence of solutions with measures as initial data and loss of mass at infinity for solutions are some of the topics treated representing a step forward in the understanding of these topics. Bennett Chow introduced a discrete model for geometric flow of a triangulated manifold.

Talks devoted to modelling included a spectacular lecture on droplet coarsening rates by Dejan Slepcev, describing his joint work with Otto and Rump. Connections to nearby problems in other contexts as chemotactic models, concentration inequalities in probability, and traffic flow were described by DiFrancesco, Gentil, and Illner. Aronson discussed the geometrical focusing corresponding to fluid wetting and eventually covering over a dry spot. Several talks were devoted to more traditional models in the kinetic theory of gases, including striking progress towards finding regular solutions by Panferov and Gamba. Novel analyses of the long time behaviour using contractive distances and entropy methods were given by Fellner, Cáceres, Carrillo and Dolbeault.

The understanding of the linearization of fast-diffusion equations and its consequences over the improvement of decay rates was another hot topic at this conference. Denzler, Cáceres, and Matthes explored aspects of this topic using different techniques. Denzler pointed out the existence of a family of explicit solutions to the porous medium equations which capture the leading, first, and second order asymptotics of the general evolution. Instead of being radially symmetric, these solutions are self-similar under affine mappings; they capture the competition between different spatial dimensions under the flow for the first time. Improved decay rates were discussed in several other contexts: reaction-diffusion equations, diffusion equations, thin film equations, logarithmic-type equations, etc., by Fellner, Kim, and Arnold. Perhaps most striking was the algebraic method described by Jüngel and Matthes for automating the search for Lyapunov entropies, and corresponding applications in existence theory and decay rates derived from functional inequalities.

Flux-limited nonlinear diffusions were shown to have uniqueness of entropy solutions by Mazón and Andreu. This family contains important examples of application as the Relativistic Heat Equation. Puel showed us how to approximate the same equation using a sequence of optimal transportation problems. Ambrosio gave a masterful lecture on convergence of the resulting Lagrangian maps in such approximation schemes. Chertock and Kurganov disussed a family of mathematical models which lead to new phenomena, such as discontinuities or evolving fronts appeared inside the support of the solution, which they exhibited numerically.

Finally, several talks were devoted to optimal mass transportation methods and geometric inequalities. Nazaret and Agueh described the use of optimal transportation methods for finding sharp constants in Sobolev and Gagliardo-Nirenberg inequalities. Sturm explained how McCann's notion of displacement convexity for a well-chosen entropy functional with respect to a transportation distance, can be used to define a notion of Ricci boundedness of a metric measure space. Such a space then automatically inherits many important analytical and geometrical properties usually associated with smooth Riemannian manifolds, such as doubling conditions, Sobolev inequalities, and Bishop-Myers-Gromov type comparisons. This resolves a major open question in geometry. Finally, Gregoire Loeper described his startling counterexample to the continuity of optimal mappings on manifolds: it shows that Ma, Trudinger, and Wang's condition guaranteeing the regularity of such maps is not only sufficient, but necessary.

4 Scientific Outcome of the Meeting

This meeting has represented a unique occasion for setting up the actual status of the research in this field and for fostering the interaction between different groups of mathematicians interested in nonlinear diffusion equations. Among these different groups we can mention mathematicians specialized in partial differential equations, geometric analysis, and the calculus of variations, and, in terms of fields of applications, people studying kinetic theory, fluid mechanics, thin films, probabilistic approaches of particle systems, plasma and solid state physics, and exotic materials.

Many new perspectives were discovered, research collaborations fostered, and directions for future investigations determined. In the words of one of the organizers: "My own research benefited tremendously. Not only did I learn the unpublished history of the line of ideas I had been pursuing, but a world of fourth order equations was opened up to me which represented new arenas of application for the second order techniques I had developed. I spent the next three months exploring consequences of what I learned at BIRS."

This workshop has showcased some of the recent progresses and set the stage for future developments, new collaborations, and cross-pollination between different communities. Certainly, the impact of this event will be measured by relevant publications in the years to come.

5 Abstracts of Talks

AGUEH, Martial (Victoria). **Sharp Gagliardo-Nirenberg inequalities and optimal transportation theory.** It is known that best constants and optimal functions of many geometric inequalities can be obtained via the optimal transportation theory. But so far, this approach has been successful for a special subclass of the Gagliardo-Nirenberg inequalities, namely, those for which the optimal functions involve only power laws. In this work, we explore the link between Optimal transportation theory and all the Gagliardo-Nirenberg inequalities. We show that the optimal functions can be explicitly derived from a specific nonlinear ordinary differential equation, which appears to be linear for a subclass of the Gagliardo-Nirenberg inequalities or when the space dimension reduces to 1. In these cases, we give the explicit expressions of the optimal functions, along with the sharp constants of the corresponding Gagliardo-Nirenberg inequalities.

AMBROSIO, Luigi (Pisa). **Convergence of iterated transport maps and nonlinear diffusion equations.** We analyze the asymptotic behaviour of iterated transport maps arising in the implicit time discretization of nonlinear diffusion equations, modelled on the porous medium equation. This analysis allows to answer affermatively to a question raised in a recent paper by Gangbo, Evans and Savin, in connection with gradient flows of a class of polyconvex energy functionals.

ANDREU Fuensanta (Valencia). **Renormalized and Weak Solutions for a Degenerate Elliptic-parabolic Problem with Nonlinear Dynamical Boundary Conditions.** We are interested in the following degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions

$$P_{\gamma,\beta}(f,g,z_0,w_0) \quad \begin{cases} z_t - \mathrm{div}\, A(x,Du) = f, \ z \in \gamma(u), \ \text{ in } Q_T :=]0,T[\times\Omega \\ \\ w_t + A(x,Du)\cdot\eta = g, \ w \in \beta(u), \ \text{ on } S_T :=]0,T[\times\partial\Omega \\ \\ z(0) = z_0 \ \text{ in } \Omega, \ w(0) = w_0 \ \text{ in } \partial\Omega. \end{cases}$$

The nonlinear elliptic operator div $A(x, Du)$ is modeled on the p-Laplacian operator

$$\Delta_p(u) = \mathrm{div}\,(|Du|^{p-2}Du),$$

with $p > 1$, γ and β are maximal monotone graphs in R^2 such that $0 \in \gamma(0)$ and $0 \in \beta(0)$. Particular instances of this problem appear in various phenomena with changes of phase like multiphase Stefan problem and in the weak formulation of the mathematical model of the so called Hele Shaw problem. Also, the problem with non-homogeneous Neumann boundary condition is included.

Under certain assumptions on γ, β and A, we prove existence and uniqueness of renormalized solutions of problem $P_{\gamma,\beta}(f,g,z_0,w_0)$ for data in L^1, and also that these renormalized solutions are weak solutions if the data are in $L^{p'}$.

ARNOLD Anton (Vienna). **Improved decay rates for the large time behavior of parabolic equations.** It is well known that the solution to heat equation behaves for large time like the Gaussian with the same moments of order 0, 1, and 2. And the "distance" from the solution to this Gaussian can be measured or estimated conveniently in terms of the relative entropy. Surprisingly, the known estimates can be improved by using the relative entropy of the initial function with respect to a Gaussian with a smaller second moment.

ARONSON Donald G. (Minneapolis). **Some aspects of the focusing problem for the porous medium equation.** I will discuss the role of self-similar solutions in the analysis of the focusing problem for the porous medium equation.

CÁCERES Maria José (Granada). **Long time behavior of linearized fast diffusion equations using a kinetic approach.** We study the long time behavior of linearized fast diffusion equations showing that their rate of convergence towards the self–similar solution can be related to the number of moments of the initial datum that are equal to the moments of the self–similar solution at a fixed time. As a consequence, we find an improved rate of convergence to self–similarity in terms of a Fourier based distance between two solutions.

The key idea to prove the results is the asymptotic equivalence of a collisional kinetic model of Boltzmann type with a linear Fokker-Planck equation with nonconstant coefficients, for which the recovering of the rate of decay in terms of the Fourier based distance is immediate. (Joint work with G. Toscani)

CARRILLO José A. (Barcelona). **Tanaka Theorem for Inelastic Maxwell Models.** We show that the Euclidean Wasserstein distance is contractive for inelastic homogeneous Boltzmann kinetic equations in the Maxwellian approximation and its associated Kac-like caricature. This property is as a generalization of the Tanaka theorem to inelastic interactions. Even in the elastic classical Boltzmann equation, we give a simpler proof of the Tanaka theorem than the ones by Tanaka (1978) and Villani (2002). Consequences are drawn on the asymptotic behavior of solutions in terms only of the Euclidean Wasserstein distance.

CHERTOCK Alina (Rayleigh). **Strongly Degenerate Parabolic Equations with Saturating Diffusion.** We first consider a nonlinear diffusion equation used to describe propagation of thermal waves in plasma or in a porous medium, endowed with a mechanism for flux saturation, which corrects the nonphysical gradient-flux relations at high gradients. We study the model both analytically and numerically, and discover that in certain cases the motion of the front is controlled by the saturation mechanism. Instead of the typical infinite gradients, resulting from the linear flux-gradients relations, we obtain a discontinuous front, typically associated with nonlinear hyperbolic phenomena. We prove that if the initial support is compact, independently of the smoothness of the initial datum inside the support, a shock discontinuity at the front forms in a finite time, and until then the front does not expand.

Adding a nonlinear convection enhances the conditions for a breakdown. In fact, the most interesting feature is the effect of criticality, that is, unlike small amplitude solutions that remain smooth at all times, large amplitude solutions may develop discontinuities. This feature is easily seen via the analysis of traveling waves: while small amplitude kinks are smooth, in large amplitude kinks part of the upstream-downstream transition must be accomplished via a discontinuous jump (subshocks). Thus induced discontinuities may persist indefinitely since the traveling waves represent a forced motion. Unlike the classical Burgers case, here, due to the saturation of the diffusion flux, the viscous forces have a bounded range. When the inertial forcing exceeds a certain threshold, the disparity between the inertial and dissipative forces is resolved by formation of a discontinuity.

CHOW Bennett (San Diego). **Combinatorial Curvature Flows.** We will discuss geometric flows of both simplicial surfaces and polygons in the plane. The combinatorial Ricci flow of surfaces takes triangulated surfaces which are piecewise hyperbolic, euclidean, or spherical and tries to make the curvatures at the vertices constant. It is related to Thurston's circle packing metrics. At the moment, very little seems to be known about combinatorial flows of planar polygons. We start with a linear equation which can be analyzed.

DASKALOPOULOS Panagiota (New York). **Type II collapsing of maximal Solutions to the Ricci flow.** We consider the initial value problem $u_t = \Delta \log u$, $u(x,0) = u_0(x) \geq 0$ in R^2, corresponding to the Ricci flow, namely conformal evolution of the metric $u(dx_1^2 + dx_2^2)$ by Ricci curvature. It is well known that the maximal solution u vanishes identically after time $T = \frac{1}{4\pi}\int_{R^2} u_0$. We provide upper and lower bounds on the geometric width of the solution and on the maximum curvature. Using these estimates we describe precisely the Type II collapsing of u at time T: we show the existence of an inner region with exponentially fast collapsing and profile, up to proper scaling, a soliton cigar solution, and the existence of an outer region of persistence of a logarithmic cusp. This is the only Type II singularity which has been shown to exist, so far, in the Ricci Flow in any dimension.

DENZLER Jochen (Knoxville). **Delocalized source type solutions for fast diffusion and porous medium.** We describe a family of explicit solutions to the porous medium and fast-diffusion equations, which are not radially symmetric, and we study their asymptotic behavior. Simarly as for the Barenblatt solution, there is reason to hope that these new solutions shed light on the asymptotics for general solutions to PME and FDE. This is joint work with Robert McCann.

DI FRANCESCO Marco (Aquila). **The Keller-Segel model for chemotaxis with prevention of overcrowding: linear vs nonlinear diffusion..** We shall discuss the effects of linear and nonlinear diffusion in the large time asymptotic behavior of the Keller-Segel model of chemotaxis prevention of overcrowding. In the linear diffusion case we provide several sufficient condition for the diffusion part to dominate and yield decay to zero of solutions. We also provide an explicit decay rate towards self–similarity. Moreover, we prove that no stationary solutions with positive mass exist. In the nonlinear diffusion case we prove that the asymptotic behavior is fully determined by whether the diffusivity constant in the model is larger or smaller than the threshold value $e = 1$. Below this value we have existence of non-decaying solutions and their convergence (along subsequences) to stationary solutions. For $e > 1$ all compactly supported solutions are proved to decay asymptotically to zero, unlike in the classical models with linear diffusion, where the asymptotic behavior depends on the initial mass.

DOLBEAULT Jean (Paris). **Nonlinear diffusions as diffusion limits of kinetic equations with relaxation collision kernels.** At the kinetic level, it is easy to relate the parameters with simple physical quantities, but the price to pay is the high dimensionality of the phase space. On the other hand, hydrodynamical equations or parabolic models are in principle simpler to compute, but their direct derivation is far less intuitive. This motivates the study of hydrodynamic or diffusion limits and in our approach, local or global Gibbs states will be considered as basic input for the modeling. This is a very standard assumption for instance in semiconductor theory when one speaks of Fermi-Dirac distributions, or when one considers polytropic distribution functions in stellar dynamics. It is the purpose of this work to provide a justification of nonlinear diffusions as limits of appropriate simple kinetic models.

Let us mention that in astrophysics, power law Gibbs states are well known (see, *e.g.*, [1], and [7] for some mathematical properties of such equilibrium states).

In our approach [2], [3] we say nothing about the physical phenomena responsible for the relaxation towards the local Gibbs state and, on the long time range, towards the global Gibbs state. We introduce at the kinetic level a caricature of a collision kernel, which is simply a projection onto the local Gibbs state with the same spatial density, thus introducing a local Lagrange multiplier which will be referred to as the pseudo Fermi level.

We prove existence and uniqueness of solutions to the kinetic model under the assumption of boundedness of the initial datum and prove the convergence to a global equilibrium. With the parabolic scaling we rigorously prove the convergence of the solutions to a macroscopic limit using compensated compactness theory. Most notably, we are able to reproduce non-linear diffusion equations $\partial_t \rho = \Delta(\rho^m) + \nabla \cdot (\rho \nabla V)$, ranging from porous medium equation to fast diffusion, $0 < m < \frac{5}{3}$, as macroscopic limits by employing the appropriate energy profiles.

In the mathematical study of diffusion limits for semiconductor physics, more results are known, starting with [4],[5]. Other reference papers are [6] and [8].

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FELLNER Klemens (Vienna). **Entropy Methods for Systems Combining Diffusion and Nonlinear Reaction.** Reaction-diffusion systems and coagulation and fragmentation of polymers are examples of models which combine diffusion and nonlinear reactions in terms of an entropy (free energy) functional. We present entropy methods (i.e. the idea how a functional inequality relates the entropy relative to equilibrium with the entropy-dissipation accounting the conserved quantities) to study global existence and long-time behaviour.

In a first part, we discuss in particular a reaction-diffusion system modelling four chemical substances with individual diffusion coefficients, which react by reversible mass-action kinetics within a bounded domain. In this case - up to our knowledge - global L^∞ bounds are unknown, but for which, at least in 1D, a polynomially growing L^∞ bound can be established due to the decay of the entropy. We improve the existing theory in 1D by getting 1) almost exponential convergence in L^1 to the steady state via a precise entropy-entropy dissipation estimate, 2) an explicit global L^∞ bound via interpolation of a polynomially growing H^1 bound with the almost exponential L^1 convergence, and 3), finally, explicit exponential convergence to the steady state in all Sobolev norms.

In a second part, we present work in progress on the Aizenman-Bak model of coagulating and fragmenting polymers with non-degenerate size-dependent diffusion coefficients. Again in 1D, we prove a-priori estimates which show immediately smoothing in time and space while in size-distribution solutions are decaying faster than any polynomial. Moreover, we are very positive to be able to establish a sharp enough entropy entropy-dissipation estimate, which will imply - similar to the strategy above - explicit exponential convergence towards the steady state.

GAMBA Irene (Austin). **Self-similar asymptotics for generalized non-linear kinetic Maxwell models.** We study long time dynamics to solutions of initial value problems to a rather general multi-linear kinetic models of Maxwell type which may describe qualitatively different processes in applications, but have many features in common. In particular we focus in the existence, uniqueness and asymptotics to self-similar (or dynamical scaling) solutions. We use a relationship of spectral properties of the problem in Fourier space to the existence and asymptotic behavior of the solution of the original initial value problem as well as the characterization of the domain of attraction to self-similar states. In particular we show that the self-similar asymptotic dynamics imply that the solutions of these type of problems evolve to "infinitely divisible“ process from the probabilistic viewpoint, where the tails and time decay laws are classified from the spectral properties related to the original problem.

Examples are models of Maxwell type in classical space homogeneous, elastic or inelastic Boltzmann equation, and the elastic Boltzmann equation in the presence of a thermostat, all with finite or infinite initial energy, as well as Pareto distributions models in economy, and Smoluckowski type of equations.

This is work in collaboration with A. Bobylev and C. Cercignani.

GENTIL Ivan (Paris). **About modified logarithmic Sobolev inequalities and applications to concentration inequalities**

GUALDANI Maria Pia (Austin). **Discontinuous Galerkin method for dissipative quantum models.** The motion of a particle ensemble interacting with an environment can be described with a Wigner approach, where the interaction mechanisms are taken into account by a Fokker-Planck scattering term. Solutions to such kind of models are characterized by an oscillatory behavior; a modified Discontinuous Galerkin method based on non-polynomial function space is used for the numerical approximation to this problem. The choice in the scheme of trigonometric functions for the finite element space allows for a better approximation to the highly oscillatory solutions.

ILLNER Reinhard (Victoria). **Modelling traffic flow using kinetic theory**

JUENGEL Ansgar (Mainz). **Algorithmic derivation of entropy-entropy dissipation inequalities by solving polynomial decision problems.** The proof of analytical and numerical properties of solutions to nonlinear evolution equations is usually based on appropriate a priori estimates and monotonicity properties of Lyapunov functionals, which are called here entropies. These estimates can be shown by subtle integration by parts. However, such proofs are usually skillful and not systematic. In this talk a systematic method for the derivation of a priori estimates for a large class of nonlinear evolution equations of even order in one and several variables with periodic boundary conditions is presented. This class of equations contains the thin-film equations, for instance.

The main idea is the identification of the integrations by parts with polynomial manipulations. The proof of a priori estimates is then formally equivalent to the solution of a decision problem known in real algebraic geometry, which can be solved algorithmically. The method also allow us to prove the non-existence of entropies and to derive new logarithmic Sobolev inequalities.

KIM Yong Jung (Taejon). **Potential comparison and long time asymptotics of convection, diffusion and p-Laplacian in one space dimension.** Recently a potential comparison technique has been developed for solutions to a nonlinear diffusion equation. This method can be applied to other problems after a suitable modification. In this talk this technique will be discussed for the cases in the title. The convergence order of the magnitude of the solution itself is shown in L^1 norm when the three terms are together. Convergence order $1/t$ is shown when only one of them exists under extra conditions for the initial value.

KURGANOV Alexander (Louisiana). **Effects of Saturating Diffusion.** I will talk about strongly degenerate parabolic PDEs with a saturating diffusion flux. The simplest model is:

$$u_t = Q(u_x)_x, \qquad (1)$$

where Q is a bounded increasing function. Such a nonlinear diffusion is "weaker" than the linear one present in the "standard' heat equation,

$$u_t = u_{xx}.$$

The effect of the saturating diffusion in (1) is manifested in a possible "delayed diffusion" phenomenon: initial discontinuities may be smeared out only after a certain (finite) time.

In the past 10 years, Philip Rosenau (Tel-Aviv University) and I together with several of collaborators of ours have been studying various effects of saturating diffusion on convection-diffusion equations,

$$u_t + f(u)_x = [u^n Q(u_x)]_x, \quad n \geq 0,$$

porous media type equations,

$$u_t = [u^n Q(u_x)]_x, \quad n > 0,$$

and reaction-diffusion equations,

$$u_t = Q(u_x)_x - f(u).$$

We have obtained several interesting, sometimes rather surprising results, and I will present some of them, including the most recent ones.

LAURENÇOT Philippe (Toulouse). **Convergence to steady states for a one-dimensional viscous Hamilton-Jacobi equation with Dirichlet boundary conditions.** The convergence to steady states of solutions to the one-dimensional viscous Hamilton-Jacobi equation $\partial_t u - \partial_x^2 u = |\partial_x u|^p$, $(t,x) \in (0,\infty) \times (-1,1)$ with homogeneous Dirichlet boundary conditions is investigated for $p \in (0,1)$. For that purpose, a Liapunov functional is constructed by the approach of Zelenyak (1968). Instantaneous extinction of $\partial_x u$ on a subinterval of $(-1,1)$ is also shown for suitable initial data.

LEE Ki Ahm (Seoul). **Geometric properties in elliptic and parabolic problems.** In this talk, we are going to discuss the geometric properties in parabolic flows, for example porous medium equations, parabolic p-Laplace equations, and free boundary problems. And the study of the asymptotic behavior of these flows will give us another promising method to find the geometric properties of solutions in elliptic problems

LOEPER Grégoire (Lyon). **Regularity of maps solutions of optimal transportation problems.** Given two probability measures μ, ν and a cost function $c(x,y)$, one seeks to minimize

$$\int c(x, T(x)) d\mu(x)$$

among all maps T that push forward μ onto ν. This work is concerned with the continuity of the minimizers. Based on the Monge-Kantorovitch duality, the minimizers are expressed though the gradient of a "c-convex" potential ϕ (c-convexity being the appropriate generalization of convexity for general cost

c instead of $c(x,y) = |x-y|^2$). This potential will solve a Monge-Ampère type equation of the form $\det(M(x,\nabla\phi) + D^2\phi) = f(x,\nabla\phi)$. Ma, Trudinger and Wang found a sufficient condition on the cost function so that for smooth positive measures, the optimal T is smooth. I will show that this condition is actually a necessary condition for regularity, and that it is equivalent to the connectedness of the c-subdifferential of c-convex functions. Finally, I will show that when the Ma, Trudinger and Wang condition is satisfied in a strict sense, one can obtain continuity of the optimal T (i.e. C^1 regularity for the potential ϕ) under lower requirements than what is needed for the usual Monge-Ampère equation $\det D^2\phi = f$.

MATTHES Daniel (Mainz). **Two applications of an algebraic method for entropy construction.** This short presentation outlines two recent extensions and applications of the algebraic method for the construction of entropy functionals as introduced by Jüngel and the speaker.

First, a variant of the method is used to estimate the rate of entropy dissipation in the logarithmic fourth order (DLSS) equation in arbitrary space dimensions.

Second, a particular (linear) Fokker-Planck equation is considered. Although the Bakry-Emery-criterion fails in this example, the algebraic approach still yields entropy dissipation estimates. These estimates give rise to a family of Beckner-type interpolation inequalities. Explicit values for the appearing constants are calculated.

MAZON RUIZ Jose M. (Valencia). **Finite Propagation Speed for Limited Flux Diffusion Equations.** To correct the infinite speed of propagation of the classical diffusion equation Ph. Rosenau proposed the tempered diffusion equation

$$u_t = \nu \operatorname{div} \left(\frac{uDu}{\sqrt{u^2 + \frac{\nu^2}{c^2}|Du|^2}} \right). \tag{7}$$

Equation (7) was derived by Y. Brenier by means of Monge-Kantorovich's mass transport theory and he named it as the *relativistic heat equation.* We prove existence and uniqueness of entropy solutions for the Cauchy problem for the quasi-linear parabolic equation

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, Du), \tag{8}$$

where $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$ and f being a function with linear growth as $\|\xi\| \to \infty$, satisfying other additional assumptions. In particular, this class includes the relativistic heat equation (7) and the flux limited diffusion equation

$$u_t = \nu \operatorname{div} \left(\frac{uDu}{u + \frac{\nu}{c}|Du|} \right) \tag{9}$$

used in the theory of radiation hydrodynamics.

We study the evolution of the support of entropy solutions of relativistic heat equation. For that purpose, we give comparison principles between sub-solutions (or super-solutions) and entropy solutions of the Cauchy problem and then using suitable sub-solutions and super-solutions, we establish the following result.

"Let C be an open bounded set in $\mathbf{R}^N$. Let $u_0 \in (L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N))^+$ with support equal to $\overline{C}$. Assume that given any closed set $F \subseteq C$, there is a constant $\alpha_F > 0$ such that $u_0 \geq \alpha_F$ in F. Then, if $u(t)$ is the entropy solution of the Cauchy problem for the equation (7) with u_0 as initial datum, we have that

$$\operatorname{supp}(u(t)) = \overline{C} \oplus \overline{B_{ct}(0)} \quad \text{for all } t \geq 0."$$

NAZARET Bruno (Paris). **Optimal Sobolev trace inequalities on the half space.** Using a mass transportation method, we study optimal Sobolev trace inequalities on the half space and prove a conjecture made by Escobar in 1988 about the minimizers.

PANFEROV Vladislav (Hamilton). **Strong solutions of the Boltzmann equation in one-dimensional spatial geometry.** We study the nonlinear Boltzmann equation in the setting of one-dimensional (plane wave) solutions, in the assumption of bounded microscopic collision rate, satisfying certain cutoffs. Using the estimates of the relative entropy and of the quadratic functional introduced by Bony we show that the "strong" bounds ensuring L^1 stability propagate globally in time.

PUEL Marjolaine (Toulouse). **A mass transport approach for a relativistic heat equation.** We present in this talk a discrete scheme for a relativistic equation obtained following the method of Jordan Kinderlehrer Otto.

SLEPCEV Dejan (Los Angeles). **Coarsening in thin liquid films.** Thin, nearly uniform, layers of some liquids can destabilize under the effects of intermolecular forces. After the initial phase, the liquid breaks into droplets connected by an ultra-thin liquid film. As the droplets interchange mass, the configuration of droplets coarsens over time. The characteristic distance between droplets and their average size grow, while their number is decreasing.

This physical process can be modeled by an equation for the height of the fluid — the thin-film equation. The evolution is a gradient flow, that is the steepest descent in an energy landscape. I will describe how information on the geometry of the energy landscape yields a rigorous upper bound on the coarsening rate.

The mass exchange between droplets can be mediated by two mechanisms: exchange through the connecting ultra-thin layer and droplet collisions. I will discuss the relative importance of the two mechanisms.

This is joint work with Felix Otto and Tobias Rump.

STURM Karl-Theodor (Bonn). **Optimal Transportation and Ricci Curvature for Metric Measure Spaces.** We introduce and analyze generalized Ricci curvature bounds for metric measure spaces (M, d, m), based on convexity properties of the relative entropy $Ent(.|m)$. For Riemannian manifolds, $Curv(M, d, m) \geq K$ if and only if $Ric_M \geq K$ on M. For the Wiener space, $Curv(M, d, m) = 1$.

One of the main results is that these lower curvature bounds are stable under (e.g. measured Gromov-Hausdorff) convergence.

Moreover, we introduce a curvature-dimension condition CD(K, N) being more restrictive than the curvature bound $Curv(M, d, m) \geq K$. For Riemannian manifolds, CD(K, N) is equivalent to $\mathrm{Ric}_M(\xi, \xi) \geq K \cdot |\xi|^2$ and $\dim(M) \leq N$.

Condition CD(K, N) implies sharp version of the Brunn-Minkowski inequality, of the Bishop-Gromov volume comparison theorem and of the Bonnet-Myers theorem. Moreover, it allows to construct canonical Dirichlet forms with Gaussian upper and lower bounds for the corresponding heat kernels.

VÁZQUEZ Juan Luis (Madrid). **Log-diffusion of measures.** We discuss the diffusion of Dirac measures surrounded by a locally integrable distribution according to the log-diffusion equation in two space dimensions. The point masses trickle into the medium at a rate of 4π units per unit time and mass location.