Classification of smooth 4-manifolds

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1 Overview of the Field

Despite spectacular advances in defining invariants for simply-connected smooth and symplectic 4-dimensional manifolds and the discovery of important qualitative features about these manifolds, we seem to be retreating from any hope to classify simply-connected smooth or symplectic 4-dimensional manifolds. The subject is rich in examples that demonstrate a wide variety of disparate phenomena. Yet it is precisely this richness which, at the time of our work at BIRS, gives us little hope to even conjecture a classification scheme.

2 Recent Developments and Open Problems

Below is a list of operations that are effective constructing and altering the smooth structure on a given 4-dimensional smooth manifold. The open problem is to determine if this is the complete list.

Surgery on a torus.; This operation is the 4-dimensional analogue of Dehn surgery. Assume that X contains a homologically essential torus T with self-intersection zero. Let N_T denote a tubular neighborhood of T. Deleting the interior of N_T and regluing $T^2 \times D^2$ via a diffeomorphism $\varphi : \partial(T^2 \times D^2) \to \partial(X - \text{int } N_T) = \partial N_T$ we obtain a new manifold X_φ, the result of surgery on X along T. The manifold X_φ is determined by the homology class $\varphi_*[\partial D^2] \in H_1(\partial(X \smallsetminus N_T); \mathbf{Z})$. Fix a basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial(X \smallsetminus N_T); \mathbf{Z})$, then there are integers p, q, r, such that $\varphi_*[\partial D^2] = p\alpha + q\beta + r[\partial D^2]$. We sometimes write $X_\varphi = X_T(p, q, r)$. It is often the case that $X_\varphi = X_T(p, q, r)$ only depends upon r, e.g. T is contained in a cusp neighborhood, i.e. α and β can be chosen so that they bound vanishing cycles in $(X - \text{int } N_T)$. We will sometimes refer to this process as a *generalized logarithmic transform* or an r*-surgery* along T.

If the complement of T is simply connected and $t(X) = 1$, then $X_\varphi = X_T(p, q, r)$ is homeomorphic to X. If the complement of T is simply connected and $t(X) = 0$, then X_φ is homeomorphic to X if r is odd, otherwise X_φ has the same c and χ_h but with $t(X_\varphi) = 1$.

Knot surgery. This operation is the 4-dimensional analogue of sewing in a knot complement along a circle in a 3-manifold. Let X be a 4-manifold which contains a homologically essential torus T of self-intersection 0, and let K be a knot in S^3. Let $N(K)$ be a tubular neighborhood of K in S^3, and let $T \times D^2$ be a tubular neighborhood of T in X. Then the knot surgery manifold X_K is defined by

$$X_K = (X \smallsetminus (T \times D^2)) \cup (S^1 \times (S^3 \smallsetminus N(K))$$

The two pieces are glued together in such a way that the homology class $[\text{pt} \times \partial D^2]$ is identified with $[\text{pt} \times \lambda]$ where λ is the class of a longitude of K. If the complement of T in X is simply connected, then X_K is homeomorphic to X.

Fiber sum. This operation is a 4-dimensional analogue of sewing together knot complements in dimension 3, where a knot in dimension 4 is viewed as an embedded surface. Assume that two 4-manifolds X_1 and X_2 each contain an embedded genus g surface $F_j \subset X_j$ with self-intersection 0. Identify tubular neighborhoods N_{F_j} of F_j with $F_j \times D^2$ and fix a diffeomorphism $f : F_1 \to F_2$. Then the fiber sum $X = X_1 \#_f X_2$ of (X_1, F_1) and (X_2, F_2) is defined as $X_1 \smallsetminus N_{F_1} \cup_\varphi X_2 \smallsetminus N_{F_2}$, where φ is $f \times$ (complex conjugation) on the boundary $\partial(X_j \smallsetminus N_{F_j}) = F_j \times S^1$. We have

$$(c, \chi_h)(X_1 \#_f X_2) = (c, \chi_h)(X_1) + (c, \chi_h)(X_2) + (8g - 8, g - 1)$$

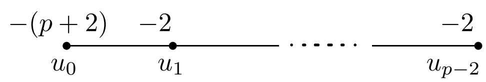
Also $t(X_1 \#_f X_2) = 1$ unless F_j is characteristic in X_j, $j = 0, 1$.

Branched covers. A smooth proper map $f : X \to Y$ is a *d-fold branched covering* if away from the critical set $B \subset Y$ the restriction $f|X \smallsetminus f^{-1}(B) : X \smallsetminus f^{-1}(B) \to Y \smallsetminus B$ is a covering map of degree d, and for $p \in f^{-1}(B)$ there is a positive integer m so that the map f is $(z, x) \to (z^m, x)$ in appropriate coordinate charts around p and $f(p)$. The set B is called the *branch locus* of the branched cover $f : X \to Y$. In the case of *cyclic* branched covers, i.e. when the index-d subgroup $\pi_1(X \smallsetminus f^{-1}(B)) \subset \pi_1(Y \smallsetminus B)$ is determined by a surjection $\pi_1(Y \smallsetminus B) \to \mathbf{Z}_d$, and B is a smooth curve in Y, then $e(X) = d\,e(Y) - (d-1)\,e(B)$ and $\sigma(X) = d\,\sigma(Y) - \frac{d^2-1}{3d}B^2$, and it follows that

$$(c, \chi_h)(X) = d(c, \chi_h)(Y) - (d-1)e(B)(2, \frac{1}{4}) - \frac{(d^2-1)}{3d}B^2(3, \frac{1}{4})$$

Blowup. This operation is borrowed from complex geometry. Form $X \# \overline{\mathbf{CP}}^2$, where $\overline{\mathbf{CP}}^2$ is the complex projective plane $\mathbf{CP}^2$ with the opposite orientation.

Rational blowdown. Let C_p be the smooth 4-manifold obtained by plumbing $p-1$ disk bundles over the 2-sphere according to the diagram



Then the classes of the 0-sections have self-intersections $u_0^2 = -(p+2)$ and $u_i^2 = -2$, $i = 1, \dots, p-2$. The boundary of C_p is the lens space $L(p^2, 1-p)$ which bounds a rational ball B_p with $\pi_1(B_p) = \mathbf{Z}_p$ and $\pi_1(\partial B_p) \to \pi_1(B_p)$ surjective. If C_p is embedded in a 4-manifold X then the rational blowdown manifold $X_{(p)}$ of [FS1] is obtained by replacing C_p with B_p, i.e., $X_{(p)} = (X \smallsetminus C_p) \cup B_p$.

Connected sum. Another operation is the *connected sum* $X_1 \# X_2$ of two 4-manifolds X_1 and X_2. We call a 4-manifold *irreducible* if it cannot be represented as the connected sum of two manifolds except if one factor is a homotopy 4-sphere. Keep in mind that we do not know if there exist smooth homotopy 4-spheres not diffeomorphic to the standard 4-sphere S^4 and that we have very little understanding of the uniqueness of connect sum decompositions of a reducible 4-manifold.

Prior to this meeting we understood that knot surgery on a given smooth 4-manifold X was obtained via a sequence of logarithmic transformations on null-homologous tori in X. A problem considered during this meeting, but not resolved, was to determine if two homeomorphic simply-connected smooth 4-manifolds are related via a sequence of log transformations on null-homologous tori.

3 Scientific Progress Made during the Research in Teams meeting

As a focal point for the start of our our meeting we concentrated on smooth 4-manifolds with small Euler characteristic. In the past few years there has been significant progress on the problem of finding exotic smooth structures on the manifolds $P_m = \mathbf{CP}^2 \# m\,\overline{\mathbf{CP}}^2$. The initial step was taken by Jongil Park, [P], who found the first exotic smooth structure on P_7, and whose ideas renewed the interest in this subject. Peter Ozsvath and Zoltan Szabo proved that Park's manifold is minimal [OS], and Andras Stipsicz and Szabo used a technique similar to Park's to construct an exotic structure on P_6 [SS]. Shortly thereafter, the organizers of this meeting produced a new method for finding infinite families of smooth structures on P_m, $6 \le m \le 8$ [FS3], and Park, Stipsicz, and Szabo showed that our techniques can be applied to the case $m = 5$ [PSS].

One goal of this meeting was to better understand the underlying mechanism which produces infinitely many distinct smooth structures on P_m, $5 \le m \le 8$. All these constructions start with the elliptic surface $E(1) = P_9$, perform a knot surgery using a family of twist knots indexed by an integer n [FS2], then blow the result up several times in order to find a suitable configuration of spheres that can be rationally blown down [FS1] to obtain a smooth structure on P_m that is distinguished by the integer n. During this meeting we explained how this can be accomplished by surgery on nullhomologous tori in a manifold R_m homeomorphic to P_m, $5 \le m \le 8$. In other words, we found a nullhomologous torus Λ_m in R_m so that $1/n$-surgery on Λ_m preserves the homeomorphism type of R_m, but changes the smooth structure of R_m in a way that depends on

n. Presumably, R_m is diffeomorphic to P_m, but we have not yet been able to show this in general. Our hope is that by better understanding Λ_m and its properties, one will be able to find similar nullhomologous tori in P_m, for $m < 5$.

In addition we developed a technique to construct interesting 4-manifolds called *reverse engineering*. The idea here is to start with a smooth 4-manifold X with non-trivial Seiberg-Witten invariants and with non-trivial first betti number. In the case of complex surfaces, such manifolds are called *irregular* surfaces. The goal would then be to find tori with trivial normal bundle with the property that the inclusion induced homomorphism on H_1 has kernel at most **Z**. In this case there is a log transform on this torus that results in a manifold X' that has betti number one less than that of X. We showed that X' has infinitely many distinct smooth structures. As a test, we applied this construction to the product of a genus two surface with itself and the symmetric product of a genus three surface. In the first case there results infinitely many distinct smooth manifolds with the same integral homology as $S^2 \times S^2$ and in the second case infinitely many distinct smooth manifolds with the same integral homology as P_3.

Concerning the problem to determine if two homeomorphic simply-connected smooth 4-manifolds are related via a sequence of log transformations on null-homologous tori we made further progress and developed a new surgical technique to alter smooth structures that will be developed in further work of the organizers.

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