The Path Partition Conjecture for Oriented Graphs

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1 Overview of the Field

The vertex set and arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively, and the number of vertices in a digraph D is denoted by $n(D)$. A *directed* cycle (path, walk) in a digraph will simply be called a cycle (path, walk). A graph or digraph is called *hamiltonian* if it contains a cycle that visits every vertex, *traceable* if it contains a path that visits every vertex, and *walkable* if it contains a walk that visits every vertex.

A digraph D is called *strong* (or *strongly connected*) if every vertex of D is reachable from every other vertex. Thus a digraph D of order bigger than 1 is strong if and only if it contains a closed walk that visits every vertex. A maximal strong subdigraph of a digraph D is called a *strong component* of D and a maximal walkable subdigraph of D is called a *walkable component* of D.

A longest path in a digraph D is called a *detour* of D. The order of a detour of D is called the *detour order* of D and is denoted by $\lambda(D)$.

If (a, b) is a pair of positive integers, a partition (A, B) of the vertex set of a digraph D is called an (a, b)-*partition* if $\lambda(D\langle A\rangle) \leq a$ and $\lambda(D\langle B\rangle) \leq b$.

If a digraph D has an (a, b)-partition for every pair of positive integers (a, b) such that $a + b = \lambda(D)$, then D is called λ-*partitionable*. The Directed Path Partition Conjecture (DPPC) states:

DPPC: Every digraph is λ-partitionable.

A directed version of the PPC that is perhaps stronger than the DPPC was stated by Bondy [2]. (His conjecture requires $\lambda(D\langle A\rangle) = a$ and $\lambda(D\langle B\rangle) = b$ while the DPPC only requires $\lambda(D\langle A\rangle) \leq a$ and $\lambda(D\langle B\rangle) \leq b$.)

Results supporting the DPPC appear in [1], [8] and [11]. The analogous conjecture for undirected graphs, known as the Path Partition Conjecture (PPC), was first formulated in 1981 and is still an open problem. (Cf [3]-[9], [12], [13] for results supporting the PPC.) A digraph D is called *symmetric* if for every $xy \in E(D)$ the arc yx is also in $E(D)$. The PPC is, obviously, equivalent to the conjecture that every symmetric digraph is λ-partitionable.

An *oriented* graph is a digraph with no cycle of length 2. We can therefore obtain an *oriented* graph D from a graph G by assigning a direction to each edge of G. We call such a digraph D an *orientation* of G. The DPPC restricted to oriented graphs states:

OPPC: Every oriented graph is λ-partitionable.

The DPPC implies both the PPC and the OPPC. We do not know the relationship between the OPPC and the PPC.

2 Recent Developments and Open Problems

The *detour deficiency* of a digraph D is defined as $p(D) = n(D) - \lambda(D)$. A digraph is *p-deficient* if its detour deficiency is p. These concepts have analogous definitions for graphs. The PPC has been proved for all graphs with detour deficiency $p \leq 3$, and for $p \geq 4$ it has been proved for all p-deficient graphs of order at least $10p^2 - 3p$ (see [3], [9]). Moreover, it is shown in [10] that if a graph G is 1-deficient or 2-deficient, then even the weaker condition $a + b = \lambda(G) - 1$ guarantees that G is λ-partitionable.

For oriented graphs the situation is very different. The OPPC has not even been settled for 1-deficient oriented graphs. Moreover, it is shown in [15] that, for every $p \geq 0$ and every pair $a, b \geq 5$, there exists a strong, p-deficient oriented graph D such that $a + b = \lambda(D) - 1$ and D has no (a, b)-partition. Thus, if the OPPC is true, it will be *best possible* in a very strong sense. These observations underline the importance of settling the OPPC for 1-deficient oriented graphs. We call this special case of the conjecture the OPPC(1).

M. Nielsen suggested a new approach for solving the OPPC(1). He defined an oriented graph D to be *k-traceable* for some k, $1 \leq k \leq n$ if every induced subdigraph of D of order k is traceable. A similar concept can be defined for graphs. It is readily seen that if a graph G of order n is k-traceable for some $k \in \{2, 3, \ldots, \lceil \frac{n}{2} \rceil\}$, then G is hamiltonian. This is not the case for oriented graphs. In fact, for every $n \geq 6$ we can construct a nonhamiltonian oriented graph of order n that is k-traceable for every $k \in \{5, 6, \ldots, n\}$. We nevertheless suspect that the following conjecture is true.

Traceability Conjecture (TC): Let D be an oriented graph of order n which is k-traceable for some $k \in \{2, 3, \ldots, \lceil \frac{n}{2} \rceil\}$. Then D is traceable.

The OPPC(1) can now be formulated as follows.

OPPC(1): If D is a 1-deficient oriented graph and $a + b = \lambda(D)$, then D is not $(a+1)$-traceable or D is not $(b+1)$-traceable.

It is clear from the above formulation that the truth of the TC would imply the truth of the OPPC(1).

3 Outcome of the Research in Teams Workshop

The focus of the workshop was to approach the OPPC(1) by proving the TC for certain classes of oriented graphs. We showed that if D is a nontraceable oriented graph that is traceable for some $k \in \left\{2, \ldots, \left\lceil \frac{n(D)}{2} \right\rceil \right\}$, then D is walkable and D is a spanning subdigraph of an MNT oriented graph D^* with the same number of strong components as D. We investigated the structure of walkable MNT oriented graphs, and this enabled us to show that the TC need only be considered for oriented MNT graphs having a very special structure. In fact, we showed that proving the TC reduces to showing that if D is an MNT oriented graph with at most three strong components, of which one is non-hamiltonian and the others are tournaments, then for each $k \in \left\{2, \ldots, \left\lceil \frac{n(D)}{2} \right\rceil \right\}$, D is not k-traceable. Using this approach, we proved that the TC, and hence the OPPC(1), holds for oriented graphs with sufficiently small or sufficiently large minimum degree, as well as for oriented graphs whose nontrivial strong components are all hamiltonian.

A paper containing the results of our workshop is in progress. Moreover, the ground work has been laid for future research. Investigating the structure of MNT oriented graphs is in itself an interesting problem, and the Traceability Conjecture for Oriented graphs is an intriguing new conjecture.

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