Report on Research Accomplished

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The problem of partial unconditionality, namely does there exist a K so that given any weakly null normalized sequence (x_n) in a Banach space and a $d > 0$, some subsequence is Elton unconditional for d with constant K, touches many areas of mathematics as evidenced in our original proposal. One such area is that of discrete approximation: given a normalized sequence, when can one be assured that every element of the space can be well approximated by a finite linear combination of the elements with coefficients chosen from a discrete alphabet? It is easy to see that the unit vector basis of c_0 has this property. In fact this property played a key role in W.T.Gowers' proof that c_0 satisfies "the ultimate Ramsey property". However other bases also possess this property as witnessed by use of the sigma-delta algorithm in signal processing applied to the summing basis. We were able to verify however that an unconditional basis with the discrete approximation property which is Elton unconditional must be the c_0 basis. This then led us to a study of such bases. Here are the main results we achieved.

Throughout, X will denote a separable infinite-dimensional Banach space and (e_i) will denote a semi-normalized basis of X. Our definitions and results can also be formulated for more general dictionaries, but for the sake of simplicity we will only consider bases.

Definition 1. A seminormalized sequence (e_i) has the (ε, δ)-*Coefficient Quantization Property* (abbr. (ε, δ)-CQP) if for every $x = \sum_{i\in E} a_i e_i \in X$ (where E is a finite subset of $\mathbb{N}$) there exist $n_i \in \mathbb{Z}$ $(i \in E)$ such that

$$\|x - \sum_{i\in E} n_i \delta e_i\| \le \varepsilon. \tag{1}$$

(e_i) has the CQP if (e_i) has the (ε, δ)-CQP for some $\varepsilon > 0$ and $\delta > 0$.
(b) A dictionary (e_i) has the (ε, δ)-*Net Quantization Property* (abbr. (ε, δ)-NQP) if for every $x \in X$ there exist a finite subset $E \subset \mathbb{N}$ and $n_i \in \mathbb{Z}$ $(i \in E)$ such that

$$(2) \qquad \|x - \sum_{i \in E} n_i \delta e_i\| \le \varepsilon.$$

(e_i) has the NQP if (e_i) has the (ε, δ)-NQP for some $\varepsilon > 0$ and $\delta > 0$.

Note that in the definition of the NQP we do not insist that vectors are approximated by vectors with the same support.

Theorem 2. *Suppose that $c_0 \hookrightarrow X$ and that X has a basis. Then X has a normalized, bounded basis which has the $(\varepsilon, c\varepsilon)$-CQP for all $\varepsilon > 0$, where c is an absolute constant (independent of X and ε).*

Theorem 3. *Let (e_i) be a semi-normalized basic sequence with the CQP. Then (e_i) has a subsequence that is equivalent to the unit vector basis of c_0 or to the summing basis of c_0.*

Theorem 4. *Let (e_i) be a normalized monotone basis for a Banach space E. Given $\eta > 0$ there exists a Banach space U with a normalized monotone basis (u_i) with the following properties:*

(a) *(u_i) has the $(\varepsilon, \varepsilon/3)$-NQP;*

(b) *there exists a subsequence (u_{n_i}) of (u_i) that is $(1+\eta)$-equivalent to (e_i).*

Theorem 5. *Let (e_i) be a seminormalized basis with the NQP. Then every subsequence of (e_i^*) has a further subsequence equivalent to the unit vector basis of ℓ_1.*

The aforementioned results have been written up in [1].

In addition we began to study the same problem for frames. Indeed a frame workshop at BIRS during our FRG provided us with an opportunity to learn about frames and deduce certain things.

Since vectors in Banach space do not have unique representation with respect to frames it turned to be much harder to formulate results similar to Theorems 2, 3, 4 and 5 as the following example shows.

Example 6. There is a tight frame (x_i) of normalized vectors (actually the union of two orthonormal bases) in a separable Hilbert space H so that for any $\varepsilon > 0$ there is a $\delta > 0$ so that if $x \in H$ there is a sequence $(n_i) \in \mathbb{Z}$, having finite support (i.e. the set $\{i \in \mathbb{N} : n_i \neq 0\}$ is finite)

$$\Big\|x - \sum_{i=1}^{\infty} n_i \delta x_i\Big\| < \varepsilon.$$

This example shows that one needs to state the quantization problem for frames differently.

Question 7. Assume H is a separable Hilbert space. Is there a normalized frame (x_i) of H with the following property:

There is a constant C and for any $\varepsilon > 0$ there is a $\delta > 0$ so that for any $x \in H$ there is a family $(n_i) \subset \mathbb{Z}$, having finite support, so that

a) $\max_{i\in\mathbb{N}} |n_i| \le C\|x\|$
b) $\big\|x - \sum_{i=1}^{\infty} n_i \delta x_i\big\| < \varepsilon$.

We were able to prove in several special cases, that such a frame does not exist, but the general problem is still open.

References

[1] [DOSZ] S. Dilworth, E. Odell, Th. Schlumprecht, and A. Zsak, *The Coefficient Quantization Property*, preprint.