Random Sorting Processes (06frg501)

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April 22 – May 6, 2006

1 Overview of the Field

The purpose of the focussed research group was to study the Uniform Sorting Network and related random models. These random models bring together ideas from several fields, including probability, interacting particle systems, algebraic combinatorics, Young tableaux and group theory.

Our main object of study was the Uniform Sorting Network; that is, a uniformly random reduced word factorization of the permutation $(n, \ldots, 2, 1)$ in the symmetric group S_n using the adjacent transpositions $(i\ i+1)$ as generators. The study of the combinatorial properties of sorting networks has been a popular area of research among algebraic combinatorialists; see the papers [8], [4], [6], [5]. The introduction of a probabilistic element in [1] has brought to light very interesting connections with the theory of interacting particle systems and with the geometry of polytopes, and a wealth of striking conjectures. It was with these developments in mind that we decided to organize a meeting to bring together experts from several different related fields and try to attack some of the open problems mentioned in [1].

2 Recent Developments and Open Problems

The central open problem in the field is the conjecture that, when viewed as a path on the permutohedron (a natural embedding into R^n of the Cayley graph of S_n generated by adjacent transpositions), the Uniform Sorting Network resembles half of a great circle in the $(n-2)$-sphere, asymptotically as $n \to \infty$. This conjecture is known to imply several remarkable statements about the behavior of random sorting networks, notably that the particle trajectories are approximately sine curves, that the density of particles in the halfway permutation is given by "Archimedes measure" on the disk, and that the swap locations satisfy a law of large numbers with intensity given by a semicircle law. We have proved this last statement in [1] using the recent work of Pittel and Romik [7] on random Young tableaux, thus offering some circumstantial evidence in support of the sphere geodesic conjecture. Other circumstantial and heuristic evidence, as well as overwhelming numerical data, also support the conjecture as a natural limiting law for the behavior of random sorting networks, and perhaps other models.

In addition to the sphere geodesic conjecture, which at the moment seems to be a difficult problem, the problem of deriving other rigorous results on sorting networks appears challenging but not impossible. Some important steps forward in this direction were made during the workshop – see below.

Other directions of research include analyzing other random sorting processes. In [1] the Uniform Transposition Sorting Network, a natural directed random walk on the permutohedron, is analyzed successfully. Some open questions remain.

3 Scientific Progress Made

BIRS provided the perfect setting for the focussed research group, and we made some important progress in the study of random sorting networks. On a broad level, it was extremely useful to bring together the experts in the field and discuss the main issues in a focussed way. The central problem of the sphere geodesic conjecture "proved its worth by fighting back", and we were not able to solve it. However, a number of major advances were made, as detailed below. In addition, the meeting enabled us to hugely increase our understanding of the problem, and a number of very promising approaches have been identified. It seems very likely that further progress will follow.

Specifically, the following advances were made:

- **"Octagon bounds".** We have proved a family of bounds on the location of the particles in a random sorting network as a function of time. Specifically, with high probability as $n \to \infty$, there are no particles outside a certain polygonal region. The region is illustrated below in the case of the half-way permutation (along with the conjectural sharp "circle bound" which would follow from the sphere geodesic conjecture mentioned above). Our argument also gives bounds on the particle trajectories, and shows that when suitably scaled, the trajectories are continuous, and in fact Hölder$_{1/2}$. The results are proved by analyzing the Edelman-Greene [4] algorithm for constructing a random staircase Young tableau from a random sorting network. We are hopeful that the same ideas may yield further bounds on the permutations and trajectories.

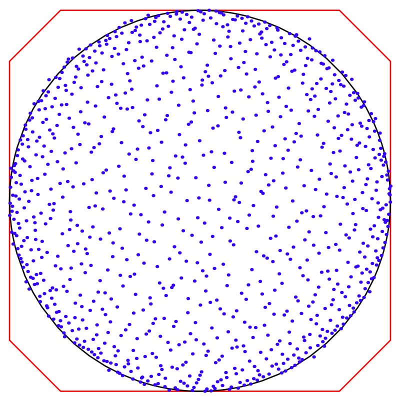

Figure 1: The half-way permutation of a random sorting network, the conjectural circle limit shape and the octagon bound.

- **Local structure.** A crucial step towards proving the sphere geodesic conjecture seems to be gaining a good understanding of the "local" structure of a random sorting network; that is the behavior of the wiring diagram over small regions of space-time. This local behavior is known to be stationary in time, but it would seem *a priori* that it might be different in different places along the spatial axis. A major step forward in our understanding was the realization that in fact the behavior should be the same in all places along the spatial axis, up to a scaling of the rate of swaps. This realization comes from rephrasing the problem in terms of staircase shape random Young tableaux. One may rigorously formulate this by saying that the swaps in a local region converge in distribution to a limiting process of swaps, and that this limit is the same for all spatial locations. We are currently working on a proof of this statement, which is likely to be completed in the near future.

Other progress made during the meeting includes the following. Several specific families of permutations have been ruled out as half-way permutations, in the sense that their probability is very small. A new and more efficient algorithm for exact simulation of random sorting networks was developed, resulting in simulations up to $n = 10000$, and even more compelling evidence for the main conjecture. We also explored, and gained some understanding into, a certain group of matrices defined using the Jacobi orthogonal polynomials that seems to exhibit the same limiting behavior as the random sorting networks.

4 Outcome of the Meeting

Several papers are in preparation: [1] was already in preparation, and has been improved as a result of the meeting; [2] and [3] are based on advances made during the meeting. It seems likely that more progress will follow soon.

References

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