Applications of torsors to Galois cohomology and Lie theory

V. Chernousov (University of Alberta),
D. Harari (École Normale Superiéure, Paris),
S. Kumar (University of North Carolina, Chapell Hill),
Arturo Pianzola (University of Alberta)

April 23 – April 28, 2005

A BRIEF INTRODUCTION

The idea of building mathematical structures out of local data has been a cornerstone of both modern Mathematics and Physics. Manifolds, distributions, simplicial complexes, vector bundles, and homogeneous spaces attest to this fact. The mathematical tools that measures the obstruction preventing us from gluing local data in a compatible way are the various cohomology theories.

In the middle of the last century the theory of algebraic varieties was establishing itself as a invaluable tool that allowed "geometric methods" to be applied to arithmetical questions. But already A. Weil had explicitly singled out that one of the most powerful classical tools, namely the construction of the quotient of a manifold by the action of a Lie group (homogeneous spaces), had no successful analogue for algebraic groups acting on varieties. (The reason being that the Zariski topology of a variety, which plays the role of the classical topology for a manifold, is too weak: there are too few open sets to trivialize actions, and these sets are too big). The answer to this riddle came from the work of Serre and of Grothendieck. The resulting theory of principal homogeneous spaces (Torsors for short), hinges around endowing schemes with the étale topology, and using various theories of "descent" to produce a coherent cohomology theory to go with it.

Several of the fundamental problems in algebra and number theory are related to the problem of classifying G-torsors and in particular of computing the Galois cohomology $H^1(k, G)$ of an algebraic group G defined over an arbitrary field k. The study of Galois cohomology is still in its early stages and many natural questions and long standing conjectures are still open. During the past two decades new insight into this theory began arising under the influence of algebraic geometry and algebraic K-theory. We note that new possibilities provided by algebraic K-theory still only begin to manifest themselves in full strength.

It has also recently become apparent that torsors can also be used to understand affine Kac-Moody Lie algebras and groups and superconformal algebras. It is possible, but at this point not known, that these methods could extend to a more general class of Lie algebras (Extended Affine Lie Algebras) around which there is today a considerable amount of interest.

Exploring the connections between these two aspects of torsors: The algebraic Geometry on one hand, and the infinite dimensional Lie theory on the other, was one of the purposes of the meeting.

PARTICIPANTS

E. Bayer-Fluckiger (Lausanne), G. Berhuy (Nottingham), S. Berman (Saskatchewan), S. Cai (UCLA), V. Chernousov (Edmonton), J-L Colliot-Thélène (Paris 11), C. Cunningham (Calgary), G. Favi (Basel), S. Garibaldi (Atlanta), Ph. Gille (Paris 11), S. Gille (Zurich), D. Harari (ENS), R. Jardine (London, Ontario), N. Karpenko (Lens), M.Knus (Zurich), S. Kumar (Chapell Hill), D. Krashen (Yale), B. Kunyavskii (Bar-Ilan, Israel), O. Mathieu (Lyon), A. Merkurjev (UCLA), J. Minac (Western Ontario), E. Neher (Ottawa), I. Panin (Petersburg), R. Parimala (Bombay), E. Peyre (Grenoble), A. Pianzola (Edmonton), A. Queguiner-Mathieu (Paris 13) M. S. Raghunathan (Bombay), W. Raskind (south California) Z. Reichstein (Vancouver), B. Remy (Lyon), A. Retakh (MIT), D. Saltman (Texas), C. Scheiderer (Duisburg), J-P Serre (College de France), R. Sujatha (Bombay), T. Szamuely (Budapest), J-P Tignol (Belgium), B. Totaro (Cambridge), A. Vishik (Moscow).

SUMMARIES OF TALKS

G–forms and cohomological invariants
by E. Bayer–Fluckiger (EPFL Lausanne, Switzerland)

Let k be a field of characteristic $\neq 2$. Milnor's conjecture, recently proved by Voevodsky, provides a classification of quadratic forms over k up to isomorphism. This gives hope for progress in related questions, for instance the classification of quadratic forms invariant by a finite group.

Let G be a finite group. One of the natural examples of G–forms is given by trace forms of G–Galois algebras. If L is a G–Galois algebra, let us denote by q_L its trace form. Let q_0 be the unit G–form – if we denote by L_0 the split G–Galois algebra, then we have $q_{L_0} = q_0$. If G has odd order, then it is known that $q_L \simeq_G q_0$ (where $\simeq_G$ denotes G–compatible isometry). If the 2–Sylow subgroups of G are elementary abelian of rank r, then in a joint paper with J–P. Serre we give a complete criterion for the isomorphis of the trace forms of two G–Galois algebras in terms of an r–dimensional mod 2 cohomological invariant.

Let us denote by $W(k)$ the Witt ring of the field k, and let $I = I(k)$ be the ideal of even dimensional quadratic forms. Let d be the 2–cohomological dimension of k. Let L and L' be two G–Galois algebras. Then Milnor's conjecture implies that if $\phi \in I^d$, then the quadratic forms $\phi \otimes q_L$ and $\phi \otimes q_{L'}$ are isomorphic. Philippe Chabloz recently proved that these forms are actually isomorphic as G–forms. Going futher in this direction, note that Milnor's conjecture implies that if $\phi \in I^{d-1}$ and if we denote by $e_{d-1}(\phi)$ its cohomological invariant, then $\phi \otimes q_L \simeq \phi \otimes q_{L'}$ if and only if $e_{d-1}(\phi) \cup d(q_L) = e_{d-1} \cup d(q_{L'})$. This talk presented some partial generalisations of this fact. One can define a notion of G–*discriminant* for q_L, denoted by $d_G(q_L)$. It is then natural to conjecture that $\phi \otimes q_L$ and $\phi \otimes q_{L'}$ are isomorphic as G–forms if and only if $e_{d-1}(\phi) \cup d_G(q_L) = e_{d-1} \cup d_G(q_{L'})$. This is known in some cases, by the work of Chabloz, Monsurro, Parimala, Schoof and the author.

Essential dimension of homogeneous forms

by G. Berhuy (Nottingham University, UK)

The essential dimension of an algebraic structure is roughly the minimal number of independent parameters needed to describe it up to isomorphism. This notion has been defined first by Reichstein an Buhler for Galois extensions of given group G in a more geometric way, then extended to any G-torsor by Reichstein (where G is an algebraic group defined over an algebraically closed field of characteristic 0).

In this talk, we compute the essential dimension of the generic homogeneous polynomial of degree d in n variables when the *g.c.d.* of n and d is a (possibly trivial) prime power. For this, we define a new numerical invariant attached to G-torsors in a geometric way, namely the canonical dimension. We then relate the canonical dimension of a certain GL_n/μ_d-torsor to the essential dimension of the generic homogeneous polynomial, and we use the properties of canonical dimension to compute it.

The algebraic connective K-theory

by S. Cai (UCLA, USA)

By using the Brown-Gersten-Quillen spectral sequence, we give a simple definition of the algebraic connective K-theory, the universal homology theory overriding the K-homology (chow groups) and algebraic K-theory. The definition of a homology theory (a Borel-Moore functor) is verified, and standard properties are proved. Relations with K-homology and K-theory are explored.

Groupe de Picard et groupe de Brauer des compactifications lisses d'espaces homogènes, I et II

by J-L. Colliot-Thélène (Université Paris-Sud, France)
et B. È. Kunyavskiĭ (Bar-Ilam University, Israel)

Soit k un corps de caractéristique nulle, $\overline{k}$ une clôture algébrique de k, et $g = \mathrm{Gal}(\overline{k}/k)$. Soient G un k-groupe connexe et X/k un espace homogène de G. Le *stabilisateur géométrique*, c'est-à-dire le groupe d'isotropie d'un $\overline{k}$-point de $\overline{X} = X \times_k \overline{k}$ est bien défini à $\overline{k}$-isomorphisme non unique près. On *note* $\overline{H}$ ce groupe. Supposons le groupe $\overline{H}$ connexe. Il y a alors un k-tore T naturellement associé au G-espace homogène X, tel que $\overline{T}$ soit le plus grand quotient torique $\overline{H}^{\mathrm{tor}}$ de $\overline{H}$. Soit X_c une k-compactification lisse de X. La $\overline{k}$-variété $\overline{X}_c$ est unirationnelle, le groupe de Picard $\mathrm{Pic}(\overline{X}_c)$ est un g-module continu discret $\mathbf{Z}$-libre de type fini et le groupe $\mathrm{Br}(\overline{X}_c)$ est fini. On note $\mathrm{Br}_1(X_c)$ le noyau de l'application de restriction $\mathrm{Br}(X_c) \to \mathrm{Br}(\overline{X}_c)$. Le quotient du groupe de Brauer $\mathrm{Br}_1(X_c)$ par l'image du groupe $\mathrm{Br}(k)$ est un sous-groupe du groupe fini $H^1(g, \mathrm{Pic}(\overline{X}_c))$.

À tout g-module continu discret M et tout entier naturel i on associe le groupe

$$\mathrm{Sha}^i_\omega(k, M) = \mathrm{Ker}[H^i(g, M) \to \prod_h H^i(h, M)],$$

où h parcourt les sous-groupes fermés procycliques de g.

Théorème A *Soient k un corps de caractéristique nulle, G un k-groupe linéaire connexe, X une k-variété espace homogène de G, de stabilisateur géométrique connexe. Soit X_c une k-compactification lisse de X.*

(i) *Le g-module* $\mathrm{Pic}(\overline{X}_c)$ *est un g-module flasque, c'est-à-dire que pour tout sous-groupe fermé $h \subset g$, on a $H^1(h, \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}(\overline{X}_c), \mathbf{Z})) = 0$, soit encore* $\mathrm{Ext}^1_h(\mathrm{Pic}(\overline{X}_c), \mathbf{Z}) = 0$.

(ii) *Pour tout sous-groupe fermé procyclique $h \subset g$, on a $H^1(h, \mathrm{Pic}(\overline{X}_c)) = 0$.*

(iii) *Soit T le k-tore associé au G-espace homogène X, et soit $\hat{T}$ son groupe des caractères. Si G est un groupe linéaire quasitrivial, i.e. extension d'un k-tore quasitrivial par un k-groupe simplement connexe, alors le quotient du groupe* $\mathrm{Br}_1(X_c)$ *par l'image du groupe* $\mathrm{Br}(k)$ *s'injecte dans le groupe* $\mathrm{Sha}^1_\omega(k, \hat{T})$*, et est isomorphe à ce dernier groupe si $X(k) \neq \emptyset$ ou si k est un corps de nombres.*

Sous l'hypothèse de (iii), nous montrons comment le g-module $\mathbf{Z}$-libre de type fini $\mathrm{Pic}(\overline{X}_c)$ est déterminé, à addition près d'un g-module de permutation, par le k-tore T – en particulier il ne dépend pas du groupe quasi-trivial G.

Ce théorème est une extension naturelle de résultats connus dans le cas où $\overline{H} = 1$ (Voskresenskiĭ 1975, Colliot-Thélène et Sansuc 1976, Borovoi et Kunyavskiĭ 2004). Ces résultats furent rappelés dans le premier exposé.

Un ingrédient important de la démonstration du théorème A est le théorème suivant, pour la démonstration duquel un ingrédient essentiel nous a été suggéré par O. Gabber.

Théorème B *Soit A un anneau de valuation discrète de corps des fractions K, de corps résiduel k de caractéristique nulle. Soit G un K-groupe quasitrivial et soit E/K un G-espace homogène de stabilisateur géométrique connexe et de tore associé trivial. Soit X un A-schéma propre, régulier, intègre, dont la fibre générique contient E comme ouvert dense. Alors il existe une composante de multiplicité 1 de la fibre spéciale de X/A qui est géométriquement intègre sur son corps de base k.*

Totaro's question on zero-cycles on G_2, F_4, and E_6 torsors

by S. Garibaldi (Emory University, Atlanta, USA)

It is a natural naive question to ask: How can one tell if a collection of polynomial equations has a common solution over a given field k? A more sophisticated version of this question asks: If a variety X has a zero-cycle of degree 1, does X necessarily have a k-point, i.e., a closed point of degree 1? Various examples show that some restrictions on the variety X are necessary for a positive answer. Several people (Veisfeiler, Serre, Colliot-Thélène) have suggested hypotheses that may be sufficient to guarantee a positive answer.

In a 2004 paper, Totaro asked whether a G-torsor X that has a zero-cycle of degree $d > 0$ will necessarily have a closed étale point of degree dividing d, where G is a connected linear algebraic group. This question is closely related to several conjectures regarding exceptional algebraic groups. Totaro gave a positive answer to his question in the following cases: G simple, split, and of type G_2, type F_4, or simply connected of type E_6. Detlev W. Hoffmann and I proved that the answer is also "yes" for all groups of type G_2 and some nonsplit groups of type F_4 and E_6. We make no restrictions on the characteristic of the base field. The key tool is a lemma regarding linkage of Pfister forms.

Twisted forms of toroidal Lie algebras

by P. Gille (Université Paris-Sud, France)
jointly with A. Pianzola (University of Alberta, Canada)

The main thrust of our project is the study of Toroidal Lie algebras via cohomological methods This leads us to the theory of reductive group schemes as developed by M. Demazure and A. Grothendieck [8]. More precisely, Algebraic Principal Homogeneous Spaces (also called Torsors for short) and their accompanying non-abelian étale cohomology, arise naturally once this new point of view is taken into consideration.

Let A be a finite dimensional algebra over a field k. An *R-form* of A is an algebra L over R for which there exists a faithfully flat and finitely presented extension S/R such that

$$L \otimes_R S \simeq_S A \otimes_k S$$

(isomorphism of S-algebras).

Since $A \otimes_k S \simeq (A \otimes_k R) \otimes_R S$, the R-algebra L is nothing but an R-form (trivialized by Spec$S$ in the flat topology of Spec$R$) of the R-algebra $A \otimes_k R$. Since Spec$R$ is affine, the isomorphism classes of such R-algebras are parametrized by $H^1(R, \mathrm{Aut}_k A_R)$ (pointed set of Cěch cohomology on the flat side of Spec$R$ with coefficients on $\mathrm{Aut}_k A_R$). The group sheaf $\mathrm{Aut}_k A_R$ is in fact an affine R-group scheme (because A is finite dimensional). If $\mathrm{Aut}_k A$ is smooth (for example if char $k = 0$), then S may be assumed to be an étale cover.

Because of connections with Extended Affine Lie Algebras (EALAs for short), the case when R is a ring of Laurent polynomial in finitely many variables is of special importance (one variable corresponding the affine Kac-Moody case). For simplicity, we will restrict our attention to this special this case.

We assume henceforth that k is of characteristic 0. Fix $n \geq 0$ and let $R_n = k[t_1^{\pm 1}, \dots t_n^{\pm 1}]$. For any positive d, define $R_{n,d} = k[t_1^{\pm \frac{1}{d}}, t_2^{\pm \frac{1}{d}}, \cdots t_n^{\pm \frac{1}{d}}]$, and let $R_{n,\infty}$ be the inductive limit of all the $R_{n,d}$.

By definition, forms are trivialized in some $fppf$ extension of the base ring. In the case of Laurent polynomials, one has very precise control over the trivializing base change.

Theorem *Let A be a finite dimensional k-algebra. Every R_n-form L of A is isotrivial (i.e. trivialized by a finite étale cover of R_n). More precisely, there exist a finite Galois extension K/k and a positive integer d such that*

$$L \otimes_{R_n} (R_{n,d} \otimes_k K) \simeq_{R_{n,d} \otimes_k K} \ A \otimes_k (R_{n,d} \otimes_k K).$$

Similarly, every reductive group scheme over R_n is isotrivial.

Multiloop algebras are the quintessential examples of forms. Assume k to be algebraically closed, and fix once an for all a compatible family $(\zeta_n)_{n>0}$ in $k^\times$ of primitive roots of unity (thus $\xi_{hd}^h = \xi_d$).

We begin by introducing the ingredients needed in the definition of multiloop algebras. Let $\sigma = (\sigma_1, \dots, \sigma_\ell)$ be a commuting family of finite order automorphisms of the k-algebra A. Let m_i be the order of σ_i.

For each $(i_1, ..., i_n) \in \mathbb{Z}^n$, consider the simultaneous eigenspaces

$$A_{i_1 \dots i_n} := \{x \in A : \sigma_j(x) = \xi_{m_j}^{i_j} \text{ for all } 1 \leq j \leq n\}$$

(which of course depend only on the i_j modulo the m_j). Finally, consider the rings extension $R_n \subset R_{n,\mathbf{m}} = k[t_1^{\pm 1/m_1}, \dots, t_n^{\pm 1/m_n}]$ where $\mathbf{m} = (\mathbf{m_1}, ..., \mathbf{m_n})$. The *multiloop algebra* associated to this data is the k-algebra

$$L = L(A, \sigma) = \oplus A_{i_1 \dots i_n} \otimes t^{i_1/m_1} \dots t_n^{i_n/m_n} \subset A \otimes_k R_{n,\mathbf{m}}$$

Observe that L has a natural R_n-algebra structure. One easily verifies that $L \otimes_{R_n} R_{n,\mathbf{m}} \simeq_{R_{n,\mathbf{m}}} A \otimes_k R_{n,\mathbf{m}}$, and that $R_{n,\mathbf{m}}/R$ is free of finite rank (hence $fppf$. In fact étale and even Galois). Thus L is an R_n-form of A which is trivialized by the extension $R_{n,\mathbf{m}}/R_n$.

Let $\mathfrak{g}$ be a finite dimensional split simple Lie algebra over an algebraically closed field k of characteristic zero. In nullity 1 loop algebras provide us with concrete realization of the affine Kac-Moody algebras (a result of V. Kac). We can in fact prove a much stronger assertion: In nullity 1 *every* form of $\mathfrak{g}$ is a loop algebra. This follows from the following result of Pianzola.

Theorem *Let* $\mathbf{G}$ *be a reductive group scheme over* $R_1 = k[t_1^{\pm 1}]$. *Then* $H^1(k[t_1^{\pm 1}], \mathbf{G}) = 1$.

This result ought to be thought as a the validity of "Serre Conjecture I " for $k[t_1^{\pm 1}]$ (the usual Conjecture I, which is consequence of a Theorem of Steinberg, corresponding to the generic fiber of R_1; namely the function field $k(t_1)$).

With this in mind, we now turn our attention to the case $n = 2$ where some interesting and unexpected behaviour arises. Assume now that K is a field of dimension 2. Serre's Conjecture II asserts that $H^1(K, \mathbf{G}) = 1$ whenever $\mathbf{G}$ is a semisimple algebraic of simply connected type. At the present time, this conjecture is still open. There is however one case where the conjecture is known to hold, and this is precisely the case when $K = k(t_1, t_2)$.

By analogy with the one dimensional case, it seems inevitable to raise the following.

Question. *Let* $\mathbf{G}$ *be a semisimple group scheme over* $R = k[t_1^{\pm}, t_2^{\pm 1}]$. *Assume* $\mathbf{G}$ *is of simply connected type. Is* $H^1(R, \mathbf{G})$ *trivial? . More generally, if* $\mathbf{G}/R$ *is semisimple and* $\lambda : \widetilde{\mathbf{G}} \to \mathbf{G}$ *is its universal covering with (central) kernel* μ, *is the boundary map* $H^1(R, \mathbf{G}) \to H^2(R, \mu)$ *bijective ?*

We have shown that the the boundary map $H^1(R, \mathbf{G}) \to H^2(R, \mu)$ is always surjective. Furthermore, if $\mathbf{G}$ is split, then $H^1(R, \widetilde{\mathbf{G}}) = 1$ (and therefore the answer to the above question is positive). But somehow surprisingly however, the answer in general is negative (we have constructed an explicit counterexample, but the classification of these exotic objects seems hard). The failure seems to be directly related to anisotropic kernels.

Diagrams and torsors

by J.F. Jardine (University of Western Ontario, London, Ontario, Canada)

Maps between objects X and Y in a homotopy category can be identified with path components of a category of cocycles, in great generality. This correspondence can be used to give a simple demonstration of the identification of isomorphism classes of torsors (torsors are generalizations of principal bundles) for sheaves of groups G with maps in the homotopy category of simplicial sheaves. This identification is a homotopy theoretic classification G-torsors; this classification result has been known at this level of generality since the late 1980s, but the new proof is much simpler and more conceptual.

A G-torsor can be characterized as a sheaf X admitting a G-action for which the corresponding Borel construction $EG \times_G X$ is isomorphic to a point in the homotopy category of simplicial sheaves. More generally, for arbitrary index categories I, I-torsors are defined to be diagrams of weak equivalences which have trivial homotopy colimits. Using the machinery of Quillen's Theorem B (which is one of the main foundational results of algebraic K-theory), one can show that homotopy colimit and derived pullback together define a bijection

$$[*, BI] \cong \pi_0(I - \mathbf{Tors})$$

relating morphisms from a point to BI in the homotopy category with the set of path components of the category of I-torsors. This definition of I-torsor and the homotopy classification both exist quite generally, and specialize to definitions of higher torsors and motivic torsors with corresponding homotopy classification results.

Higher torsors can be thought of as special types of diagrams which take values in simplicial sheaves, and are defined on sheaves of categories I enriched in simplicial sets. Sheaves of groupoids enriched in simplicial sets are the objects of a homotopy theory which is equivalent to the full homotopy theory of simplicial sheaves, for which the fibrant objects represent higher stacks. The homotopy classification result for higher torsors does not depend on the theory of higher stacks, and the result for the full category of sheaves of categories enriched in simplicial sets was unexpected.

A bound for canonical dimension of the (semi-)spinor groups

by N. Karpenko (Universite d'Artois, Lens, France)

In the talk we discuss the *canonical dimension* $\mathrm{cd}(G)$ of a linear algebraic group G defined over a field F which was introduced recently by Berhuy–Reichstein. The general question raised by Berhuy–Reichstein is to determine $\mathrm{cd}(G)$ for every split simple algebraic group G.

For the spinor group, representing a particularly difficult case of the above general question, one knows that $\mathrm{cd}\,(\mathrm{Spin}_{2n+1}) = \mathrm{cd}\,(\mathrm{Spin}_{2n+2})$, so that we will discuss only $\mathrm{cd}(\mathrm{Spin}_{2n+1})$ here.

Although the canonical dimension of, say, a smooth projective variety X can be expressed in terms of algebraic cycles on X, there are no general recipes for computing $\mathrm{cd}(X)$ or $\mathrm{cd}(G)$. A better situation occurs with the *canonical p-dimension* cd_p, a "p-local version" of cd, where p is a prime: a recipe for computing $\mathrm{cd}_p(G)$ of an arbitrary split simple G is obtained by Merkurjev and Karpenko. In particular, one has

$$\mathrm{cd}_2(\mathrm{Spin}_{2n}) = \frac{n(n-1)}{2} - 2^l + 1 ,$$

where l is the minimal integer such that $2^l \geq n+1$ (and for any odd prime p, one has $\mathrm{cd}_p(\mathrm{Spin}_{2n+1}) = 0$). Since $\mathrm{cd}\,(G) \geq \mathrm{cd}_p(G)$ for any G and p, we have a *lower bound* for the canonical dimension of the spinor group, given by its canonical 2-dimension.

We establish the following *upper bound* for spinor groups:

$$\mathrm{cd}\,(\mathrm{Spin}_{2n+1}) \le n(n-1)/2.$$

This result improves the previously known upper bound $n(n+1)/2$, established by Berhuy–Reichstein. The proof makes use of the theory of non-negative intersections, of duality between Schubert varieties, and of the Pieri formula for a variety of maximal totally isotropic subspaces.

Note that the lower bound for $\mathrm{cd}(\mathrm{Spin}_{2n+1})$, given by $\mathrm{cd}_2(\mathrm{Spin}_{2n+1})$, coincides with our upper bound if (and only if) $n+1$ is a power of 2. Therefore, for such n, we get the precise value: if $n+1$ is a power of 2, then

$$\mathrm{cd}(\mathrm{Spin}_{2n+1}) = \mathrm{cd}(\mathrm{Spin}_{2n+2}) = \frac{n(n-1)}{2}\ .$$

Our second main result is the following upper bound for the semi-spinor groups $\mathrm{Spin}^{\sim}_{2n+2}$, obtained by the similar technique: for any odd one has

$$\mathrm{cd}(\mathrm{Spin}^{\sim}_{2n+2}) \le n(n-1)/2 + 2^k - 1,$$

where $k = v_2(n+1)$ (the 2-adic order of $n+1$).

Importance of the spinor and semi-spinor groups in this context is explained by the fact that these groups represent the only difficult cases of the following general question: let G be a split simple algebraic group, having a unique torsion prime p (a prime p is a *torsion prime* of G if and only if $\mathrm{cd}_p(G) \ne 0$); is it true that $\mathrm{cd}(G) = \mathrm{cd}_p(G)$?

Zero cycles on homogeneous varieties

by D. Krashen (IAS, Princeton)

The study of projective homogeneous varieties and their invariants has been a source of many interesting problems and has various applications in recent years. For example, Panin's description of the algebraic K-theory of homogeneous varieties has resulted in the useful index reduction formulas of Merkurjev, Panin and Wadsworth. The study of algebraic cycles and the motives of these varieties has also played an important role in quadratic form theory, and in particular, Voevodsky's proof of the Milnor conjecture. The structure of the Chow groups and motives of these varieties continues to be an active area of research with many unresolved questions.

In the talk, we introduce tools for studying the Chow group of 0-dimensional cycles on a projective variety using results from Suslin and Voevodsky's work on algebraic singular homology. This allows us to connect the study of the group of zero cycles to studying the more geometrically naive notion of R-equivalence (i.e. connecting points with rational curves) on symmetric powers of the original variety.

We apply these ideas by showing that the symmetric powers of certain homogeneous varieties may be related to spaces which parametrize commutative étale subalgebras in a central simple algebra. To make this precise, we define moduli spaces of étale subalgebras in a central simple (or Azumaya) algebra. These spaces are very interesting in their own right, as many open questions in the area of central simple algebras concern the existence and structure of certain types of subfields in a division algebra. We show that in certain cases these moduli spaces are R-trivial, and we apply this to determining the Chow group of zero cycles for certain homogeneous varieties. This allows us to show that the Chow group of zero dimensional degree zero cycles is trivial for involution varieties as well as for certain Severi-Brauer flag varieties. This was previously known to be true for involution varieties of index no more than 2 (by work of Swan, Karpenko and Merkurjev) and for Severi-Brauer varieties (by work of Panin).

On Cachazo-Douglas-Seiberg-Witten Conjecture for simple Lie algebras

by S. Kumar (University of North Carolina at Chapell Hill, USA)

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over the complex numbers. Consider the exterior algebra $R := \wedge(\mathfrak{g} \oplus \mathfrak{g})$ on two copies of $\mathfrak{g}$. Then, the algebra R is bigraded with the two copies of $\mathfrak{g}$ sitting in bidegrees (1,0) and (0,1) respectively. To distinguish, we will denote the first copy of $\mathfrak{g}$ by $\mathfrak{g}_1$ and the second copy of $\mathfrak{g}$ by $\mathfrak{g}_2$.

The diagonal adjoint action of $\mathfrak{g}$ gives rise to a $\mathfrak{g}$-algebra structure on R compatible with the bigrading. We isolate three 'standard' copies of the adjoint representation $\mathfrak{g}$ in R^2, where R^2 is the total degree 2 component of R. The $\mathfrak{g}$-module map

$$\partial : \mathfrak{g} \to \wedge^2(\mathfrak{g}), \qquad x \mapsto \partial x = \sum_i [x, e_i] \wedge f_i,$$

considered as a map to $\wedge^2(\mathfrak{g}_1)$ will be denoted by c_1, and similarly,

$$\begin{aligned} c_2 :& \mathfrak{g} \to \wedge^2(\mathfrak{g}_2), \text{ and} \\ c_3 :& \mathfrak{g} \to \mathfrak{g}_1 \otimes \mathfrak{g}_2, \quad x \mapsto \sum_i [x, e_i] \otimes f_i, \end{aligned}$$

where $\{e_i\}_{i \le i \le N}$ is any basis of $\mathfrak{g}$ and $\{f_i\}_{1 \le i \le N}$ is the dual basis of $\mathfrak{g}$ with respect to a normalized Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}$. We denote by C_i the image of c_i.

Let J be the (bigraded) ideal of R generated by the three copies C_1, C_2, C_3 of $\mathfrak{g}$ (in R^2) and define the bigraded $\mathfrak{g}$-algebra

$$A := R/J.$$

The Killing form gives rise to a $\mathfrak{g}$-invariant $S \in A^{1,1}$ given by

$$S := \sum_i e_i \otimes f_i.$$

Motivated by supersymmetric gauge theory, Cachazo-Douglas-Seiberg-Witten made the following conjecture. They proved the conjecture for classical $\mathfrak{g}$. Subsequently, Etingof-Kac proved the conjecture for $\mathfrak{g}$ of type G_2 by using the theory of abelian ideals in $\mathfrak{b}$.

Conjecture [CDSW] *(i) The subalgebra $A^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants in A is generated, as an algebra, by the element S.*

(ii) $S^h = 0$.

(iii) $S^{h-1} \neq 0$, where h is the dual Coxeter number.

The aim of this work is to give a uniform proof of the above conjecture part (i). In addition, we give a conjecture, the validity of which would imply part (ii) of the above conjecture.

To prove part (i), we first prove that the graded algebra $B^{\mathfrak{g}}$ is isomorphic with the singular cohomology of a certain (finite dimensional) projective subvariety $\mathcal{Y}_2$ of the infinite Grassmannian $\mathcal{Y}$ associated to $\mathfrak{g}$, where $B := R/\langle C_1 \oplus C_2 \rangle$. The definition of the subvariety $\mathcal{Y}_2$ is motivated from the theory of abelian ideals in the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. This isomorphism is obtained by using Garland's result on the Lie algebra cohomology of $\hat{\mathfrak{u}} := \mathfrak{g} \otimes t\mathbb{C}[t]$; Kostant's result on the 'diagonal' cohomolgy of $\hat{\mathfrak{u}}$ and its connection with abelian ideals in $\mathfrak{b}$; and a certain deformation of the singular cohomology of $\mathcal{Y}$ introduced by Belkale-Kumar.

Steenrod operations in algebraic geometry

by A. Merkurjev (UCLA, USA)

Steenrod operations in algebraic geometry were originally defined by Voevodsky in the context motivic cohomology. P.Brosnan found an elementary definition of the Steenrod operations on the Chow groups of algebraic varieties. His definition uses equivariant Chow groups of Edidin and Graham and the construction relies on embedding to a smooth scheme.

In the talk a new direct construction of the Steenrod operations modulo 2 is presented. Namely, the Steenrod operations (of homological type) of a scheme X are defined as the Segre classes of the tangent cone of X. All the standard properties of the Steenrod operations can be proven directly.

Non-commutative version of purity

by I. Panin (Steklov Institute, St. Petersburg, Russia)

Let $\mathcal{F}$ be a covariant functor from the category of commutative rings to the category of sets. We say that $\mathcal{F}$ *satisfies purity for R* if

$$\bigcap_{\mathrm{ht}\mathfrak{p}=1} \mathrm{Im}\,[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(K)] = \mathrm{Im}\,[\mathcal{F}(R) \to \mathcal{F}(K)].$$

For certain functors $\mathcal{F}(R)$ injects into $\mathcal{F}(K)$ for all regular local rings R. In this case the purity of $\mathcal{F}$ for R implies that

$$\bigcap_{\mathrm{ht}\mathfrak{p}=1} \mathcal{F}(R_{\mathfrak{p}}) = \mathcal{F}(R) \subset \mathcal{F}(K).$$

Now we switch to a specific functor. For that consider a characteristic zero field k, a reductive algebraic k-group G (connected one) and a functor $\mathcal{F}$ which takes a commutative k-algebra R to $H^1_{\text{ét}}(R, \mathrm{G})$. We make the following conjecture:

the functor $\mathcal{F}$ satisfies the purity for regular local rings containing k.

The conjecture is a kind of extension of the known conjecture of A. Grothendieck and J.-P. Serre. They conjectured the injectivity. Here a purity is conjectured. It can be viewed as a non-commutative version of Gersten's conjecture in K-theory. In the talk we discussed in certain details this conjectures for interesting examples of reductive groups like PGL_n, $\mathrm{SL}_{1,A}$, $\mathrm{O}(q)$, $\mathrm{SO}(q)$.

Algebras of prime degree over function fields of surfaces

by R. Parimala (Tata Institute, Mumbai, India)
jointly with M. Ojanguren (Lausanne University, Switzerland)

It is an open question whether division algebras of prime degree are cyclic. Over number fields, cyclicity of all central simple algebras is a classical theorem due to Hasse-Brauer-Noether. Further the index and the exponent coincide for all division algebras over a number field. Artin raised the question whether the index and the exponent coincide for central simple algebras over a C_2-field. Artin's question is answered in the affirmative for function fields of surfaces over an algebraically closed field of characteristic zero by de Jong. We explain a method of proof of cyclicity of prime degree algebras over such fields using de Jong's techniques.

Tori in quasi-split groups

by M. S. Raghunathan (Tata Institute, Mumbai, India)

In this talk a proof of the following result was outlined:

Let k be any field and G a quasi-split k-algebraic group, S a maximal k-split torus in G, $Z(S) = T$ the centraliser of S and $N(S)$ the normaliser of S. Let $W = N(S)/Z(S)$ be the Weyl group-scheme over k. Let $i : W \hookrightarrow \mathrm{Aut}(T)$ be the natural inclusion. Now k-isomorphism classes of tori of dimension l (= dimension T) are in bijective correspondence with elements of the Galois cohomology set $H^1(k, Aut(T))$. A necessary and sufficient condition that a torus B over k is realisable as a k- subtorus of G is that class $[B]$ of B in $H^1(k, Aut(T))$ be in the image of $H^1(k, W)$. This is a consequence of the following stronger assertion: let $\pi : H^1(k, N(T)) \longrightarrow H^1(k, W)$ and $i : H^1(k, N(T)) \longrightarrow H^1(k, G)$ be the natural maps. Then π maps kernel $i(= i^{-1}(\text{trivial class in } H^1(k, G))$ onto $H^1(k, W)$.

A key ingredient of the proof is the theorem of Steinberg that every regular semisimple k-conjugacy class in G contains a k-rational point.

Group-theoretic compactification of Bruhat-Tits buildings

by B. Rémy (Lyon 1, France)
jointly with Yves Guivarc'h (Rennes 1, France)

Let G be a simply connected semisimple algebraic group, defined over a non-archimedean local field F. We denote by G_F the locally compact group of its rational points, and we denote by X the Bruhat-Tits building of $G_{/F}$. We are interested in compactifying the vertices V_X of X by group-theoretic means, so that we eventually obtain structure results on the rational points G_F (i.e. parametrizations of remarkable classes of closed subgroups of G_F). We first prove convergence of some sequences of compact open subgroups of G_F in the Chabauty topology. This enables us to define the desired compactification of V_X. We obtain then a structure theorem showing that the Bruhat-Tits buildings of the Levi factors all lie in the boundary of the compactification. We obtain an identification theorem with the polyhedral compactification, previously defined by E. Landvogt. We finally prove two parametrization theorems extending the correspondence between maximal compact subgroups and vertices of X: one is about Zariski connected amenable subgroups, and the other is about subgroups with distal adjoint action.

Cyclic algebras over p-adic curves

by D. Saltman (Texas University, USA)

The study of the structure of division algebras goes back 150 years since they were first defined. The issue has always been how to describe their structure. The first examples of division algebras were so called cyclic algebras - defined simply using a cyclic Galois extension. Since then non-cyclic algebras have been found, but only with complicated (precisely composite) degree, where the degree of a division algebra is an integer describing its size. Thus in some ways the first question about division algebras is still unsolved, namely, whether all division algebras of prime degree are cyclic.

Another strain in the theory of division algebras is their study over special fields, where over time the "special" fields have gotten more and more general. This approach is best illustrated by the Hasse-Brauer-Noether-Albert theorem that all division algebras over global fields are cyclic. In this talk we discussed a "higher dimensional" special field, namely, the function field of a curve over a p-adic field. What we showed was that when q is a prime not equal to p, then any division algebra of degree q over such a field is cyclic.

Of equal importance to the actual result is the methods we used. The fields we are concerned with are best viewed as the function field of surfaces S over the p-adic integers. By a result of Grothendieck, such surfaces have Brauer group 0. What this means is that the division algebras over such surfaces are determined by their so called "ramification". As a consequence of this, showing that a division algebra is cyclic is equivalent to showing that one can "split" its ramification by a cyclic Galois extension of the right size. It turn out that another way to view this result is that it is a result on splitting ramification over surfaces, and as such it has had application to a much broader class of fields than treated here.

The arguments of Grothendieck and Tits on splitting fields
by B. Totaro (Cambridge University, UK)

One of the great achievements of mathematics is the 19th-century classification of the simple Lie groups by Killing and Cartan. There are four infinite families of groups and just five exceptional groups. Chevalley showed in 1958 that the same classification works for the simple algebraic groups over any algebraically closed field.

The classification of simple algebraic groups over an arbitrary field is much richer. It includes as a special case some of the fundamental problems of algebra, such as the classification of quadratic forms over an arbitrary field. Nonetheless, one can hope to answer basic questions such as: given a simple algebraic group of a given type over a field, what degree of field extension is needed to make it into the standard (Chevalley) group?

Using the idea of torsors, and the definition of the Chow ring of a classifying space, we give an improved proof of a theorem by Grothendieck which gives a strong connection between the classification of simple algebraic groups over arbitrary fields and the topology of the corresponding compact Lie groups.

As a result, we can do topological calculations and read off information about splitting fields. Tits solved these problems for many types of groups, but we are able to solve these problems in the remaining cases, notably for the groups E_8 and $\mathrm{Spin}\,(n)$. We find, for example, that any algebraic group of type E_8 over any field becomes isomorphic to the Chevalley group E_8 after a field extension of degree dividing 2880. The number 2880 is best possible. This is satisfying in that questions about E_8, the largest exceptional Lie group, have often been the hardest of all questions about Lie groups.