Hamiltonian systems with symmetry

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The *conservative, or Hamiltonian, dynamical systems* are finite or infinite dimensional dynamical systems that model physical phenomena on time scales over which dissipation is not dominant. In the idealized models, dissipation is absent, multiple time scales are usually present, and the long time dynamics is very delicate. The systems often exhibit temporal chaos. The subject has existed for centuries as celestial mechanics, but the current application domains (such as underwater vehicle dynamics [5][8][9], molecular dynamics and spectra [6][7][13], fluids and plasmas [1][4][10], and foundational physical field theories [2][3]) far exceed its origins.

When symmetry is present in these systems, one can seek solutions which are generated by one parameter subgroups of the symmetry group (i.e. of the form $\exp(\xi_e t)p_e$ where ξ_e is a Lie algebra element called the generator and $p_e \in P$). These are the *relative equilibria*, and they correspond to equilibria in the reduced spaces. The physical form of these solutions depends on the symmetry. For the three dimensional rotation group $SO(3)$, they will be uniformly rotation solutions, such as the circular orbit of a satellite. In the case of a neutrally buoyant underwater vehicle with coincident centers of mass and buoyancy, the symmetry is the Euclidean group $SE(3) = SO(3) \ltimes \mathbb{R}^3$, and the relative equilibria correspond to screw motions. When the underwater vehicle has an additional axial material symmetry, the system symmetry is $SE(3) \times SO(2)$, and the relative equilibria correspond to screw-spinning motions. When the symmetry group is not compact, such as the Euclidean symmetry group, establishing the stability of relative equilibria is delicate. Generally, in the noncompact case, a common criterion—formal stability—is insufficient to establish stability, and a more restrictive criterion—T_2 stability—must be used [12]. There is a gap between T_2 and formal stability. Stability inside the gap has been established using KAM theory, for certain relative equilibria of the system of an underwater vehicle [11] with coincident centers of mass and buoyancy. The T_2 stability theory and the KAM-desingularization technique, both due to workshop participants, are the state of the art in the area.

The workshop participants met to hammer out the details of an article that they are writing which considers an example where the gap actually occurs: the relative equilibria consisting of the falling, spinning motion of an axially symmetric underwater ellipsoid with non-coincident centers of mass and buoyancy, where the symmetry group is $SO(2) \ltimes \mathbb{R}^3 \times SO(2)$. The target audience for this work consists not just of mathematicians, but possibly also engineers and physicists, so it is important to find an exposition in the most basic language. Also this is important because the target audience must be brought to accepting that there is a subtle impact of this work on some highly regarded liturature, such as [5, 8, 9].

The stability problem, it was determined, can in the case of the symmetry in question, be generally addressed using a widely known, venerable, technique which reduces an abelian symmetry by eliminating "cyclic coordinates". After this reduction, one is reduced to a parameterized, two degree of freedom system. There is an additional circular symmetry at the parameter corresponding to the relative equilibrium in question, and the stability is a symmetry-breaking phenomenon from a completely integrable system. The general understanding of what typically occurs for any system that has the same symmetry group was established. Suitable coordinates are required to prove KAM stability inside the gap. These should be accessible to and recognizable by the target audience, and such coordinates were developed over the course of the workshop. A draft article was completed which the workshop participants anticipate will be rapidly completed and disseminated.

Another reason for considering the falling, spinning relative equilibria, is that it has nontrivial isotropy. A considerable amount of work on the isotropy problem has already been completed, but that was deliberately excluded from the theory developed in [12]. The workshop provided an opportunity to work again on this project. One of the base problems is to ensure that the presence of isotropy is fully taken advantage of in the stability theory. This was not so clear because isotropy implies singularities in the reduced spaces, at which methods which rely on a smooth structure are inapplicable, This problem was resolved over the course of the workshop: the isotropy is usually compact, in which case a purely topological proof was found that shows the presence the singularities in the reduced spaces will not affect the stability issue.

In this workshop, a team of three participants interacted very intensely. Progress was made on the isotropy project, and priorities sorted and possibilities found for further collaboration. A stalled project (the axisymmetric stability project) was reinvigorated, and work which would have taken many months, if it could have been completed at all, was largely completed in one week, with a far superior outcome. Partly this was due to an extensive preparation for the workshop, following an initial consultation a year earlier at the Bernoulli Institute of EPFL Switzerland, but it was also due to the excellent BIRS environment. The team is scattered across Canada and the UK, but the BIRS Research in Teams program enabled it to meet, concentrate on, and in large part resolve, a difficult problem, and to prepare the way for future collaborations.

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